678

BASIC STUDIES OF HYDRAULIC ESSENTIALS OF OPEN CHANNEL FLOWS CONTRIBUTIVE TO HYDRAULIC DESIGN OF CHANNEL STRUCTURES

水路構造物の水理学的設計法に

関する基礎的研究

June, 1959

YOSHIAKI IWASA

BASIC STUDIES OF HYDRAULIC ESSENTIALS OF OPEN CHANNEL FLOWS CONTRIBUTIVE TO HYDRAULIC DESIGN OF CHANNEL STRUCTURES

June, 1 9 5 9

YOSHIAKI IWASA

CONTENTS

INTRODUCTORY STATEMENT

The essential purpose of hydraulic engineering is to deal with the control and utilization of the natural waters of the earth by means of artificial treatments. From the ancient prehistoric days, the continuous endeavourment of the mankind has been progressed to accomplish this purpose. The diversity of the term " hydraulic engineering " serves many purposes in engineering problems. Water is necessary for domestic and industrial use. The agricultural contribution of water to increase the crop productions must be furnished with the complete irrigation system as one of the hydraulic engineering works. The generation of power production also uses water. Furthermore, water may be used as a mean of transport and as a source of recreation. In addition to controlling water for benificial uses, the hydraulic engineers may be called on to protect cities, towns and farms from excess water by flood control. The engineering improvement of hydraulic engineering, therefore, will be resulted in the promotion of human welfare and of peaceful life for mankind through the control and utilization works of hydraulic engineering.

The fruitful achievement of hydraulic works is substantially obtained by the design and construction of hydraulic works based on the essential features of water flow. Speaking in engineering description, all possible flow characteristics in design problems must be exactly predicted by means of either analytical or experimental procedures. It consequently gives a rise that the basic scientific knowledge of water flows known generally as hydraulics is of fundamental importance. Although the methods of prediction in flow characteristics are widely divergent, the most effective solution of problems encountered in hydraulic designs of works is evidently obtainable as the best feature of basic flow characteristics through all procedures of approach in basic science of hydraulics and hydrodynamics. This treatment is a modern procedure in hydraulics and characterized by the combination of physics of the flow behaviours in the light of pastt experience and mathematical interpretation made possible by recent advances in basic science.

The problems of open channel flows, which are extensively treated in the present study, have also developed since scientists and engineers established the basis of flow principles known as the one dimensional procedure. In relation to practical designs of hydraulic works, solutions are often given by the determination of surface profiles of water. The method to estimate the surface profiles of water has been progressed with the advance of study of gradually varied flows since Dupuit and Bresse did. The rapidly varied flows as counterpart of gradually varied flows are also studied by many investigators. Both theories of gradually and rapidly varied flows are extremely promoted by the critical depth theory, which will be described in this study. The flow resulted from rainfall in natural and artificial channels are essentially unsteady, so that all the problems for hydraulic design of control and utilization works must be solved as the unsteady problems. As a special case, the steady-state solution will be available for controlled discharge in particular hydraulic projects. The unsteady flow in open channels, however, is characterized by the so complicated phenomena combined with turbulence and non-linearity, so that the deduction of complete solution is still difficult. Nevertheless,there are so many problems which are reducible to quasi-steady state problems for particular purposes in engineering aspects, and therefore, the hydraulics of steady flows in open channels will furnish one of the most important branch of flow behaviours. Former studies of steady and unsteady flow behaviours concern mainly with the flow in channels of constant channel section. Nearly almost all natural channels and artificial watercourses involve continuous changes in channel geometry, grade and channel roughness, known as channel transitions and controls, so that the research works must be subject to such behaviours to furnish the real advance of hydraulics of open channel flows.

In the present study which will concern with the basic flow principles of open channel flows, the analysis of determination procedure of surface profiles in canals and the functional variety of flow characteristics by control structures, the research purpose is to reveal the transitional characteristics of flow in open channels and give available informations for the basic requirements in hydraulic design of conveyance and control structures in gravity projects.

Part I concerns with the basic flow characteristics of open channel flows in channel transitions and controls to ensure the further development of the study. In the first chapter, the basic principles of open channel flows in velocity and pressure distributions, one dimensional procedures of hydraulic analysis, hydraulic significance of boundary layer theory to open channel flows, and characteristics of a moving discontinuity in open channels will be conclusively described. In the second chapter, the general theory of transitional characteristics of steady flows in channel transitions and controls is treated. In the classical theory of gradually varied flows, which is still used, the theoretical confirmation of possible surface profiles of water is only made with the use of the basic non-linear equation of flow in uniform channels. In the classical treatment of rapidly varied flows, the rather empirical theorems of Bélanger and Böss are widely used to establish the head-discharge relationship of control structures. These theorems can be unified and are occured simultaneously by the generalized theory of Jaeger. It is still insufficient to establish the complete behaviours of steady flows. With respects to these problems, the geometric theory of non-linear ordinary differential equation will be introduced to provide the complete generalized theory of transitional characteristics of steady flows. This chapter is then the essential fundamentals of the steady state hydraulics of open channel flows, and as results of analysis, the combined universal treatment for both gradually and rapidly varied flows will be obtained, as an extensive theory of classical hydraulics developed since Dupuit and Bresse did. The third chapter concerns with the general features of transient characteristics of open channel flows to furnish the hydraulic requirements for design of conveyance structures by means of the one dimensional procedure. Especially, the transient characteristics of flows are resulted from those of translatory waves and surges which are similar in their wave pattern to the cnoidal and thus solitary, as a limiting case, waves. The emphasis is consequently put on the hydraulic behaviours of solitary waves as the second approximation of open channel flows characterized by the appreciable magnitude of vertical acceleration.

Part II will concern with the transitional characteristics of gradually varied flows in channel transitions and controls. As the hydraulic design of conveyance structures is essentially the exact determination of surface profiles of water for particular design discharges, so the theory of gradually varied flows is the basic scientific tool for hydraulic designs of conveyance structures in gravity projects. Especially, main attention in Chapter 4 is directed to the transitional characteristics of gradually varied flows and the classification of resulting surface profiles of water in channel transitions and controls. Usual procedures to estimate the surface profiles of water are the numerical analysis and graphical method. Mathematically speaking, channel transitions and controls may yield the singular point in the basic equation for open channel flows, so that much errors for calculation are involved in the solution in the immediate vicinity of a singular point, if the knowledge of transitional characteristics is ignored. The theory which will be described herein is particularly advantageous to the exact estimation of surface profiles of water, and the reservoir capacity in mountainous area and the influenced reach of back water by control structures are precisely estimated.

In Part III, the hydraulic performances of various control structures are discussed, as one of applications of transitional characteristics of rapidly and gradually varied flows. The channel control is a class of hydraulic structures in which the control section is involved. The hydraulic performance of control section is to determine the complete feature of open channel flows passing through this section. With relation to the Bélanger-Böss theorems, consequently, the head-discharge relationship is also established for given channel geometry by a single water-level measurement. The hydraulic characteristics of control section, therefore, provide the basis of discharge measurement by open channels. The first chapter describes the general feature of control structures to flow diversities. The second chapter deals with the hydraulic performance of round crested and circular weirs and hydraulic requirements for design problems as one of the basic forms of control structures. The discharge-head relationship of flows over spillway crests as an example of applications of hydraulic behaviours at control sections is followed in the third chapter. The fourth chapter concerns with the hydraulics of sharp crested weir and free overfall as control structures. Problems described herein have been investigated by many engineers since the 18 th century and are not still solved because the basic flow pattern is so complicated as to be not subjected in explicite forms. Although the present treatment by means of the transitional characteristics of curved flows can not make the behaviours definitely, it will be an effective mean for further development of analysis to hydraulics of weir flows. On the contrary to the foregoing hydraulic characteristics of rapidly varied flows, the fifth chapter deals with the hydraulics of flumes as measuring devices of discharge of gradually varied flows.

In Part IV, as the conclusive description of foregoing discussion of flow behaviours in open channels involving channel transitions and controls, the basic requirements for hydraulic designs of conveyance and control structures are treated. The treatments of design procedures are practically divided into mild and steep conveyance structures, channel transitions and controls. Being distinctly different from the design procedure of mild channels, the hydraulic design of steep channels must be considered by special hydraulic features of supercritical flows like the formation of roll waves and the entrainment of air bubbles into the flow. The transition section in a canal line also must be designed depending on its purpose. The control structure may often result in the rapid change of velocity and pressure of the flow, so that the hydraulic requirement for design purpose is added by the pressure requirements which cause sudden decrease or increase of pressure. When the conveyance and control structures are designed, the basic flow behaviours in designed channels must be definitely predicted so that their hydraulic functions serve satisfactorily to the original purposes. In reality, it is readily understood that all hydraulic characteristics described in this study will surely provide the basic essentials for hydraulic designs of conveyance and control structures.

All of the subjects in the present paper were promoted by the author with cooperation of many colleagues and students in the Hydraulics Laboratory, Department of Civil Engineering, Kyoto University, under the supervision of Dr. Tojiro Ishihara, Professor of Hydraulic Engineering and Dean of the Faculty, for recent several years. The original purpose to establish completely the design criteria for various types of conveyance and control structures in hydraulic works is not fully obtained. It is, however, seen for the author that the basic flow characteristics at channel transitions and control and the analytical procedures necessary to hydraulic designs of canals and other hydraulics structures in engineering projects are essentially established with systematic verification of experimental data, and this indeed is the true purpose of presentation of the study for the fulfillment of the degree of Doctor of Engineering.

I. BASIC FLOW CHARACTERISTICS IN CHANNEL TRANSITIONS AND CONTROLS

1. Basic Principles of Fluid Flows in Open Channels

1 - 1 - 1 Basic Remarks

Towards the end of the 19 th century the science of fluid mechanics began to develop in two directions which had practically no point in common. On the one side there was the science of theoretical hydrodynamics which was evolved from the Eulerian equation of motion for a frictionless, non-viscous fluid and which achieved a high degree of completeness. Since, however, the results of this so-called classical science of hydrodynamics stood in glaring contradiction to experimental results -- in particular as regards the very important problem of head losses in pipes and channels as well as with regard to the drag of a body which moves through a mass of fluid -- it had little importance. For this reason, practical hydraulic engineers, prompted by the need to solve the important problems arising from the rapid progress in technology, developed their own highly empirical science of hydraulics. The science of hydraulics was based on a large number of experimental data and differed greatly in its methods and in its objects from the science of theoretical hydrodynamics.

It will be, however, of evidence that the most effective solution of alomst all problems encountered in design problems of hydraulic works is obtained by the best feature combined through all methods of approach in both theoretical hydrodynamics and rather empirical classical hydraulics. This, indeed, is the modern procedure in the science of hydraulics, which is furnished by the combination of physics of the flow behaviours in the light of past experience and dynamic interpretations made possible by recent advances in modern theoretical hydrodynamics.

The history of hydraulics as scientific treatment of fluid flows has been gradually progressed with the development of human civilization as observed in old structures of canals and aqueduct and the behaviours of fluid motion in open channels have been thus studied only from the practical side.

For a long time, the principal idea of the analysis of fluid motion in classical hydraulics has been only based on the so-called Bernoulli's theorem of energy conservation. Evidently, the water flow in open channels is the Navier-Stokian fluid and its motion is characterized by the dynamic expressions of classical theoretical hydrodynamics derived by the Newtonian principles of motion. As the modern hydrodynamics, therefore, has been progressed the trend of hydrodynamic verification of classical theorems of hydraulics in open channel flows has been also prompted.

The first is an attempt to obtain the dynamic equation of open channel flows from the Navier-Stokes' and Reynolds' equations in theoretical hydrodynamics by directly integrating variables through the zone of flow. This approach was originally done by J. Boussinesq[1)], and G.H. Keulegan and G.W. Patterson[2)] refined it in 1943. The other is from the one dimensional procedures in energy and momentum conservation laws of fluid motion considered a stream flows as a single tube, which is the basis in the 19 th century hydraulics and still of great importance in the analysis of the physics of open channel flows as a refined form. The momentum approach was also initiated by de Saint Venant[3)] and extended to the non-hydrostatic flow as a second approximation characterized by the appreciable value of curvature at the free surface by Boussinesq and that derived by the energy approach also was refined by C. Fawer[4] in 1937. These procedures, however, are imposed by some ambiguous assumptions for the velocity profile and the resulting pressure distribution. Recently, F. Sérre[5] in 1953, Y. Iwagaki[6] in 1954, and the author[7] in 1955 studied the one dimensional procedures to derive the basic relationship of open channel flows and obtained the fruitful success in this field.

Recent trend of deduction of the one dimensional equations in open channel flows by means of the momentum and energy theorems in the Newtonian principles of motion is to evaluate the magnitude of errors which can result from these approximations. The surface resistance and the resulting velocity profile in the direction of running water bring the irregular distributions of energy and momentum. Such influences are estimated by introducing the correction coefficients in notations of Coriolis and Boussinesq. For the practical calculation in design problems, the uniform distribution in velocity and the hydrostatic distribution in pressure are commonly assumed and usual coefficients of Coriolis become then unity. This simplified assumption and the hydrostatic assumption in pressure lead both equations of the momentum and energy approaches to be in inequality, as seen in the later section, owing to the approximate expressions of the physical behaviours of fluid motion in both approaches. These two approaches are of equivalent characteristic if the basic physics of fluid flow is expressed in exact mathematical forms by both approaches.

A large number of experimentations in laboratories, however, indicate the essential character of gradually varied flow is given by the hydrostatic law of pressure and even in the rapidly varied flow it is still valid as an approximation, added by the curvilinear influence resulted from a curved flow. Only in problems of flow near critical regime like an efflux from a gate and others, the influence of vertical acceleration of fluid itself becomes appreciable and the resulting surface configuration is sometimes in a form of sinusoidal in a first approximation and of cnoidal in higher approximations. General characters of second approximation in flow behaviours are described by the surface curvature and the resulting non-linear equation is so complex as to permit the solution of physical behaviours of such flows.

The above description is centered on a flow in which a turbulent character generated by the surface resistance has fully developed to the free surface throughout the zone of flow. There are, however, many instances in which the rotational characters, which are actually resulted from the surface resistance along the boundary and the viscous fluid itself, are so small as to be practically ignored, and the assumption that the flow from a reservoir into a conduit or over a spillway is irrotational will often permit use of mathematical analysis with close approximation of actual behaviours, whereas the flow is influenced by the surface resistance, which is especially appreciable near the boundary. Here, the concept of the boundary layer initially proposed by L. Prandtl will be introduced in the analysis of open channel flows, though its great progress was mainly achieved in aerodynamic field. In fact, it is feasible to observe from experimentation that the boundary layer will develop to the free surface. Quite different characters of boundary layer growth in the open channel flow from those in the unconfined flow are imposed by the existence of free surface, as indicated by A. Craya and J.W. Delleur[8]. It may be, therefore, of primary importance to investigate again the basic principles of open channel flows in terms of the boundary layer theory. It is also one of the purpose of this chapter to study the flow behaviours of open channel flows in the light of knowledge in modern hydrodynamics and hydraulics recently developed.

The investigation of basic principles of open channel flows in channel transitions and controls is first accented on the velocity distribution of flows and associated boundary resistance as the most significant parameter for engineering use, followed by the thorough investigation of pressure distribution in accelerative flows. As a resulting form of preceding consideration, the basic dynamic equation of flows in open channels will also be described in forms of the one dimensional procedure. Furthermore, the basic principles of boundary layer theory of open channel flows and the growth process are indicated by means of the geometric theory of ordinary differential equation known as the topological method in non-linear mechanics, as one of the indications in the modern hydraulic research trends. Finally, the dynamics of moving shock, which is also familiar for hydraulic engineers as a bore, is followed for the further discussion of design problems in conveyance and control structures.

As a scientific theory, however, becomes more exact, so does it of necessity trend to assume a more mathematical form. This statement must not mean that the form becomes more difficult or more abstruse,but rather that, when the fundamental laws in the physics of open channel flows have reached a stage of clear formulation with the aid of systematic experiments conducted by the modern facilities and the skillful technique, useful deduction can be made by the exact process of applied mechanics.

1 - 1 - 2 Velocity Distributions and Surface Resistance

The fluid flow when it enters the chute is uniform throughout a zone of flow or irrotational in velocity profile. Later, the fluid near the solid boundary is retarded, the retarding zone spreads farther and farther to the free surface, and ultimately the final velocity distribution is obtained. This phenomenon is resulted from the momentum interchange by the surface resistance in turbulent flows, and the basic concept itself was conjectured by scientists in the 18 th century.

The subject on the formulation of close relationship between the distribution of velocity and the surface resistance is of basic significance in hydrodynamics and hydraulics, and as a matter of fact, it may be said many problems in practical hydraulics will be provided by the head losses due to the frictional resistance on the solid boundary, and much endeavours have been, therefore, accented to establish the dynamics of velocity profile and surface resistance. Great progress was achieved by the concept of Prandtl's momentum transfer, that of Kármán's similarity law and so on, though practical hydraulic engineers in the 19 th century as well as even in the present time established their own empirical relations mainly by means of field observations or the dimensional analysis.

The physics of stream flow in the hydraulics of open channel flow is expressed in a form of the one dimensional procedure, in which mean values of velocity and elevation are unknown, used as a powerful mean of analysis from the 19 th century. The establishment of relationship between the so-called frictional formulas of open channel flows like Chézy, Manning and others empirically obtained by practical engineers, and the results derived from the dynamic behaviours of turbulent flow in modern hydrodynamics is thus required for the further advance in modern hydraulics of open channel flows. Such attempts have been made by G.H. Keulegan[9), R.W. Powell[10), H. Rouse[11), and Y. Iwagaki[12) and especially

Iwagaki indicated the hydraulic significance of Manning formula in smooth and rough channels in terms of the modern knowledge of theoretical hydrodynamics. Another relationship between the velocity distribution and mean value of velocity is seen in notations of Coriolis and Boussinesq parameters.

C. Jaeger[13], following the work of Boussinesq, indicated the significant result that the establishment of interconnection between both approaches of energy and momentum in open channel flows was obtained if the evaluation of Coriolis' or Boussinesq's parameters on the dispersion of irregular distribution of energy and momentum owing to the velocity distribution was valid. It also demonstrates the range of applicability of the usual one dimensional equation of flows derived by the energy and momentum theorems, in which the hydrostatic pressure is assumed to prevail in a moving fluid. Hereupon, the secondary influence of the acceleration of a moving fluid in the upward direction is introduced in the physical behaviours of open channel flows, followed by the non-linear equation of higher order characterized by the surface curvature as Boussinesq and Fawer did. Clear establishment of such an influence will be connected in terms of the mathematical expression of stream function in classical hydrodynamics, as seen in the later section.

Of essential character in the physics of open channel flows as well as other problems in hydraulics is the distribution of velocity provided by the surface resistance or diffusion process of momentum resulted from the resistance. Although a wide variety of research for establishment of close relationship in the flow has been made with the great advance in theoretical and experimental fluid mechanics by many investigators, a final goal is not reached.

In this section, the basic knowledge on the distribution of velocity and its associated velocity formulas will be explained for the practical convenience of further study in the present purpose.

(a) Velocity Distribution in the **Flow Direction**

The change of velocity with the depth of water was revealed by old Italian hydraulic engineers in the 17 th century. In the first decade of 18 th century, Dubuat observed the velocity distribution in small canals and obtained the mean velocity formula in terms of the surface velocity. After then, many hydraulic engineers in Europe obtained their own formulas of velocity distribution, among which the Bazin formula is famous. All of the velocity formulas are substantially based on the empirical approach furnished with a number of experimental data and the knowledge of classical hydrodynamics in those days has not yet contributed to the real advance of practical science of hydraulics.

As the water flow is characterized by the Newtonian principles of motion in a form of the so-called Navier-Stokes' equation, so the velocity distribution is a solution of the basic equation in laminar flows and of the Boussinesq or Reynolds equations in turbulent flows, which are of more importance in practical hydraulics. Until the present day, however, no general methods have been available for the deduction of solutions owing to the great mathematical difficulties as a consequence of their being non-linear.

Only the way possible to approach the fruitful success in this subject is to derive the solutions under steady uniform regime. For a long time, scientists and engineers thus have been enforced to establish the velocity distribution in pipes, conduits, and channels and even in the present day, the considerable endeavours are still devoted to make the relationship clear in the light of past experimental data and knowledge of modern theoretical hydrodynamics.

Practical hydraulic engineers have developed their own method to establish the velocity distribution, which is essentially empirical and furnished by the dimensional analysis, and especially applied to the formulation of frictional losses in pipes and conduits. In the past the most popular form of velocity profile has been of the empirical power type of $u \propto y^{1/m}$, in which m varies from 4 to an indeterminate limit with increasing Reynolds number and among them the most famous formula is the 7 th power type known as the Blasius law. Keulegan[9)] and T. Tsubaki[14)] in 1947 proposed the same relations in open channel flows for the establishment of mean velocity formula. Fig. 1-1 indicates the correlationship of exponent to Reynolds number, which has been obtained by the experimental data of the Hydraulics Laboratory, Kyoto University. Within a small range of variation in flow characteristics in terms of Reynolds number, a simple formulation of the velocity distribution even in open channel flows on smooth bed will be expected. The exponent in the power law, however, is changed with the change of Reynolds number and boundary materials, and furthermore, the Froude number will also influence to the flow in open channels, so that the substantial character of physics in turbulence is not concerned in this approach, and therefore, for a

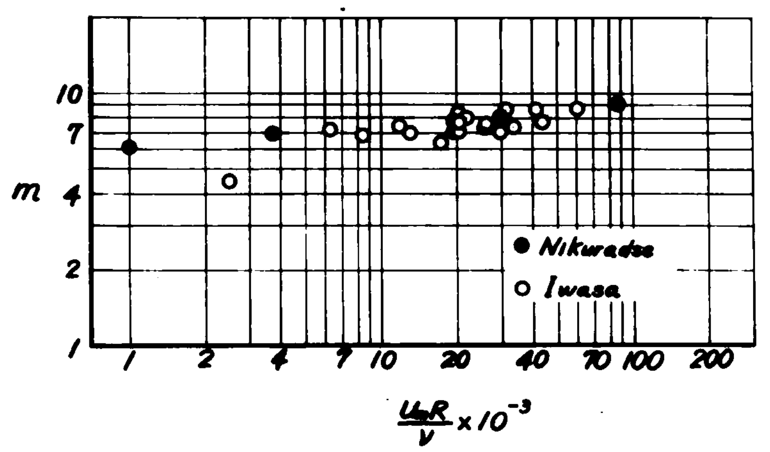

Fig. 1-1 Relationship between exponent and Reynolds number

wide range of Reynolds number, the applicability of the power law to practical calculation in problems of hydraulics is considerably limited.

The velocity distribution as a combined feature of modern hydrodynamics and experimental method in classical hydraulics is provided by a solution of the Reynolds' equation, which is in itself insufficient to make the theoretical analysis possible and in a form of

$$\rho\frac{\partial u}{\partial t} = \rho g \sin\theta + \frac{\partial}{\partial x}(p_{xx} - \rho uu - \rho u'u') + \frac{\partial}{\partial y}(p_{yx} - \rho uv - \rho u'v'), \tag{1}$$

$$p_{xx} = -p + 2\mu\left(\frac{\partial u}{\partial x}\right) \quad , \quad p_{xy} = p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

in which x and y : distances along and from the channel bottom, respectively, u and v : velocity components, ρ : density of water, t ; time, p : pressure, g : acceleration of gravity, μ : dynamic viscosity and the prime indicates fluctuated values of velocity.

Under the condition of steady uniform flow, which is the simplest case in flow pattern, Eq.(1) becomes, once integrating and putting $\tau = \tau_0$, shear along the boundary at y = 0,

$$\tau_0(1 - y/h) = \mu(\partial u/\partial y) - \rho u'v'. \tag{2}$$

Eq.(2) is still insufficient to deduce the form of velocity distribution in turbulent flows, if the accomplishment of formulation in the physics of turbulence and associated velocity fluctuation have not been attained.

Before the establishment of equation of Reynolds, Boussinesq proposed his own idea on the similarity between the molecular motion in the laminar flow and the eddy motion in the turbulent flow, essentially characterized by the Reynolds stress, permit the shear resulting from two types of motion to be described in a similar fashion. The eddy viscosity η expressed by $-\rho u'v'/(\partial u/\partial y)$, however, must be expected to vary from point to point in a manner which can be predicted only from the knowledge of turbulent characteristics, while the viscosity is the same at all points in the flow.

When the eddy viscosity is assumed constant throughout the zone of flow as an engineering approximation, Eq.(2) becomes

$$(\mu + \eta)(\partial u/\partial y) = \tau_0 (1 - y/h), \tag{3}$$

and the resulting velocity distribution is then

$$u = u_s - \{\rho g h^2 \sin\theta / 2(\mu + \eta)\}(1 - y/h)^2. \tag{4}$$

As the eddy motion is extremely predominant in turbulent flows compared with the viscous shear, so Eq.(4) indicates the theoretical verification of empirical relation obtained by Bazin.

For engineering purposes the eddy viscosity must be expressible in terms of other variables of flow and channel characteristics. In many problems involving the analysis of suspension load, the eddy viscosity is often assumed proportional to the product of u_m and h, and this assumption permits the resulting resistance formulas should be of Chézy type, while the eddy viscosity is actually dependent on channel and flow characteristics. S. Hayami[15] studied the detailed behaviours of eddy viscosity and of resulting velocity distribution with the use of the theory of turbulence by H. Gebelein[16].

Apart from the theoretical analysis of velocity distribution, a large number of experimental data obtained by many scientists and engineers indicate that the velocity profile in turbulent flow is a family of two dimensionless parameters with respect to Reynolds number and depth. The attempt to describe the velocity profile as a single parameter expression has been eagerly made and then the subject in hydraulic research to obtain a universal law of velocity distribution in the light of past experience and recent advances in theoretical hydrodynamics has also been prompted. Great progress of this subject was achieved with the introduction of mixing length theory of Prandtl in the first decade of this century, which, together with theoretical works of Karman and Taylor as well as systematic experiments by many investigators, paved the way for the theoretical clarification of the interrelationship of the Reynolds and Boussinesq parameters.

Assuming the velocity fluctuation is proportional to the gradient of mean velocity, $u' \propto v' \propto \ell(\partial u/\partial y)$, Eq.(2) becomes

$$\tau_0(1 - y/h) = \rho\{\nu + \ell^2(du/dy)\}(du/dy), \tag{5}$$

in which ν : kinematic viscosity and ℓ : mixing length.

Prandtl assumed the mixing length was proportional to the distance from the bed, $\ell = \kappa y$, and $\tau = \tau_0$, and obtained that the velocity distribution was in a form of logarithmic function of

$$(u/u^*) = A + (1/\kappa)\log(u^*y/\nu), \tag{6}$$

for smooth channels. J. Rotta[17] extended the analysis of this universal law to a considerable degree.

The application of universal law of velocity distribution to the open channel flow has been treated by many hydraulic engineers and especially Keulegan, Powell, and Iwagaki obtained many fruitful conclusions of turbulent characteristics of flow with the aid of logarithmic distribution of velocity. Of extremely different characteristic of open channel flows in velocity distribution to those of pipe flows is that the velocity profile is not a curve of one parameter of Reynolds number and thus curves of velocity distribution are shifted by the given flow characteristics. It is understood the foregoing essential difference of characteristics between the physics of turbulent flow in confined flow like open channel flows and that in pipe flows was due to the existence of free surface influence known as the hydraulic instability in the mixing length theory.

The hydraulic instability, which is one of essential characters of rapid flow and resulted from the boundary resistance, may be certainly suggested to influence the mixing length in turbulence. On the other hand, the physical character of surface disturbance as a sequence of hydraulic instability will influence the turbulence and the velocity fluctuation in the flow to some extent. The extended criterion of hydraulic instability described in the later section indicates its formation is strongly limited by the character of surface disturbance, so that it seems the establishment of universal law on velocity distribution of open channel flows in the light of modern knowledge of theoretical hydrodynamics is still far away and the possible way is furnished with the precise measurements of turbulent fluctuation and associated variations in flow characteristics of open channel flows by the experimental technique. In fact, by the recent works of F.H. Clauser[18)] on the velocity distribution in turbulent boundary layer, a complete picture of turbulent velocity profile is obtained by the combination of the outer profile characterized by the constant eddy viscosity and the inner profile characterized by the inner eddy viscosity proportional to u^*y.

(b) Resistance Formulas in Terms of Mean Velocity

Nearly all practical problems in the hydraulics of open channel flows are related to the one dimensional procedure in terms of mean values of velocity and elevation. Only one exceptional case is the problem of boundary layer growth of open channel flows. A convenient form of boundary resistance for the one dimensional analysis, therefore, is also expressed in terms of mean velocity throughout the zone of flow. Many empirical relationships between the mean velocity and the boundary resistance with past experimental data have been developed by engineers and scientists. Among them, widely used forms are of power type and especially the Chézy and Manning formulas are most famous for practical engineers.

A general form of mean velocity in terms of the power type is known as

$$u_m^a = (1/n)R^{1+b}\sin^m\theta \quad , \tag{7}$$

where, n: channel roughness and practically assumed constant for particular channel geometry and boundary, a, b and m: numerical constants, and Eq.(7) may be called as the Vedernikov power law, being derived by the Vedernikov number in hydraulic instability. When certain definite values of a, b and m derived by experimental data are inserted, the empirical formula available for the range of experimentation is obtained for particular channels. All of empirical formulas like Chézy and others are commonly established by such procedures. The universality of frictional formulas for hydraulic problems is, however, deduced by the combination of theoretical knowledge in turbulent flows and experimental data capable of furnishing the basic theory. Otherwise, the applicability of the formula is extremely limited for problems in the hydraulics of open channel flows.

The dynamic significance of the Chézy formula, which is of most convenient type of formulas for the hydraulic analysis, is derived by the assumption of constant eddy viscosity proportional to the discharge. The insertion of the assumption, $\eta = \rho k_1 u_m h$, into Eq.(3) leads to the frictional resistance as

$$u_m^2 = (g/3K_1)h\sin\theta \quad , \text{ or } \quad u_m^2 = (g/3K_1)R\sin\theta \quad . \tag{8}$$

in which, the substitution of $C^2 = (g/3k_1)$ yields the well known Chézy formula. If the above assumption is valid, the Chézy formula is of convenient type for a certain range of flow verified by the experimentation and the Chézy roughness also becomes a constant dependent on boundary materials.

The validity of foregoing results and the correlationship between the mean velocity and the characteristics of channel and flow are accomplished by a large number of experimental data. Fig. 1-2 indicates the relationship between the local drag coefficient defined by $\lambda_0 = (2g/C^2)$ and Reynolds number obtained by Powell, Iwagaki[19], and the author for smooth channels, though the actual channels are commonly rough.

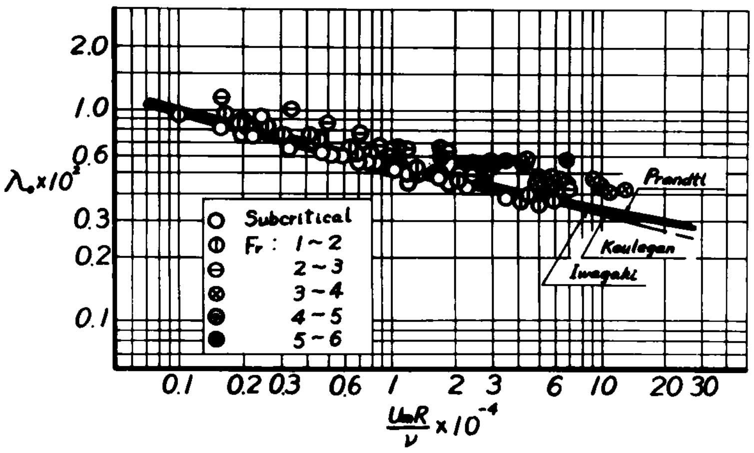

Fig. 1-2 Relationship between local drag coefficient and Reynolds number for smooth channels

Evidently, λ_0 is a function of Reynolds number, and of another significance for the correlationship in open channels is that the value of λ_0 in the rapid flow is changed, compared with that in the tranquil flow for the same Reynolds number. As Jegorow[20], M. Homma[21], Powell and Iwagaki have already indicated, it means that C is not constant but a function of Froude number as well as Reynolds number, and it will be supposed Iwagaki's proposal on the change of mixing length in the theory of Prandtl is essentially deduced from this experimental result. Formulas of Keulegan, Prandtl and Iwagaki are also indicated in the same figure, and among them Iwagaki's curve for drag coefficients expressed in terms of a function of Reynolds and Froude numbers seems most pertinent in the hydraulics of open channel flows, though it becomes small for high rapid flows.

The main purpose of hydraulic design problems in conveyance structures, which is also the essential feature of the present study, is to estimate the configuration of water surface for particular channels and a given design discharge. Nearly all channels involve channel transitions and controls, in which the local change in channel geometry and boundary resistance is involved, so that values of Reynolds and Froude numbers locally change from point to point. The local drag coefficients are thus variable with the change of significant parameters of open channel flows, even though the resistance formula established in the uniform flow is extended to the non-uniform flow as an engineering approximation. The tremendous labours for the precise estimation of drag coefficient is therefore required. The actual surface profile is obtained by the combination of local configurations calculated by a suitably estimated drag coefficient in a particular reach, in which the flow characteristics are assumed practically constant. For this reason, the influence of local change in channel geometry to the local numbers of Reynolds and Froude will be treated.

The Reynolds number is, for a constant discharge, in a form of

$$R_e = (u_m R/\nu) = (QR/\nu A). \tag{9}$$

The change of R_e is thus

$$(dR_e/R_e) = (dA/A)\{(dR/dA)s - 1\} = (M - 1)(dA/A), \tag{10}$$

introducing the following shape parameter M for channel geometry, which is in a form of

$$M = 1 - R(ds/dA) = s(dR/dA). \tag{11}$$

The change of Froude number is also obtained, in the same manner,

$$(dF_r/F_r) = -\{(2 + M)/2\}(dA/A). \tag{12}$$

Fig. 1-3 indicates the change of numbers of Reynolds and Froude for the shape parameter. In the same figure, the relation between M and the ratio of water depth to channel width, (h/b), for rectangular channels, is also indicated. For wide channels, M approaches unity and the limiting value of M indicates the flow is two dimensional. The change of Reynolds number is small for wide channels and becomes larger for narrower channels. On the contrary, the opposite tendency is seen in the behaviours of change in Froude number.

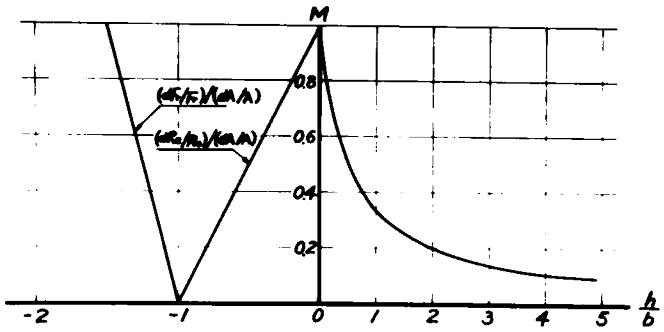

Fig. 1-3 Changes of numbers of Reynolds and Froude for channel shapes

Among the evident significances of the Manning formula as an empirical relationship of mean velocity in practical engineering problems, the change of roughness coefficient as a function of flow characteristics was small compared with that of Chézy formula has been concluded by many hydraulic engineers. The roughness coefficient of Chézy and Manning are assumed to be expressible in terms of R_e, F_r, (R/k_s) and $\sin\theta$ and they are

$$C,\ n = C,\ n(R_e,\ F_r,\ R/k_s,\ \sin\theta\ ) \tag{13}$$

The changes of roughness thus become, with the aid of the shape parameter M,

$$d(C, n)/(C, n) = -(dA/A)\Big[(1 - M)(R_e/C, n)\{\partial(C, n)/\partial R_e\} + (2 + M)/2\cdot(F_r/C, n)\{\partial(C, n)/\partial F_r\} - M\{(R/k_s)/C, n\}\{\partial(C, n)/\partial(R/k_s)\}\Big] + (\sin\theta/C, n)\{\partial(C, n)/\partial\sin\theta\}\{\partial\sin\theta/\sin\theta\}. \qquad (14)$$

As the evident characters of Manning roughness is that the change of n is rather small for a wide range of parameters of flow and channel characteristics, compared with that of Chézy roughness, so the most significant conclusion of the behaviours of Manning roughness associated with the local change of channel geometry for rough as well as smooth channel is obtained in the following.

The Manning roughness n of particular channels is practically assumed constant as an engineering approximation and the evaluation of n is uniquely determined by the boundary materials. The actual behaviours of n, however, as well as that of Chézy roughness are influenced by flow characteristics, and the foregoing conclusion will be only limited to the engineering practice. The detailed study of Manning roughness is treated by Iwagaki by means of the universal law of logarithmic velocity distribution. Nevertheless, it is still hopeless to furnish completely the behaviours of roughness coefficient by the theoretical analysis of hydrodynamics, and the precise estimation of roughness coefficient, which varies from point to point with the change of flow characteristics and channel geometry, is required for the available solution encountered in the hydraulics of open channel flows and even in practical engineering problems.

(c) Distribution of Vertical Velocity

The foregoing treatment of velocity distribution in the running water is essentially related to the steady uniform flow, in which the stream-line is parallel to the channel bed, and in the first approximation the conclusion is extrapolated to the non-uniform or unsteady flows. In this subsection, some behaviours of vertical velocity in a moving fluid, which induce the curvilinear stream line, will be concerned, though no available references have been made, owing to great difficulties even to measure the magnitude of vertical velocity in a most simplified pattern of open channel flows.

If the flow is observed from the point view of theoretical hydrodynamics, the curvature of stream line and the vertical velocity are closely related together through the equation of stream line, which is

$$(dx/u) = (dy/v). \tag{15}$$

If the slope and the curvature of stream lines and thus of free surface are appreciable, as seen in the flow near critical regime, the magnitude of vertical velocity is also expected to be extremely large. Boussinesq[1] first attempted to describe the relationship between the vertical velocity and the curvature of stream line and assumed the vertical velocity was expressible by

$$(v/u) = (y/h)\{(\partial h/\partial x) + (1/u_m)(\partial h/\partial t)\}. \tag{16}$$

F. Sérre[5] and the author[7] introduced the concept of stream function in theoretical hydrodynamics into the provision of vertical velocity and the latter indicated the assumption of Boussinesq was valid only for the uniform velocity distribution often defined as the translatory profile.

The local velocity in the running flow direction, u, is assumed as

$$u(x, y, t) = u_m(x, t)\cdot f(y/h), \tag{17}$$

in which f(y/h) is a function imposed by the velocity distribution like the power law as well as the universal law. When the power law is applied to the velocity distribution as the empirical relationship, $f(y/h)$ is also of power type and the application of the universal law to the flow leads to that $f(y/h)$ is the following expression, with the aid of the momentum approach in the one dimensional analysis,

$$f(y/h) = 1 + \sqrt{\beta - 1}(1 + \log y/h), \tag{18}$$

where, β is the momentum correction factor of Coriolis.

If the flow is completely two dimensional, the stream function is obtained by integrating Eq.(17) with respect to y, and it becomes, with the use of the equation of continuity,

$$\Psi(x, y, t) = hu_m F(y/h) + \varphi(x, t), \tag{19}$$

in which, $F(y/h)$ is an integral function of $f(y/h)$ and $\varphi(x, t)$ is an arbitrary function of x and t. Inserting the condition that the vertical velocity and $F(y/h)$ are zero at the channel bed, the final expression of vertical velocity becomes

$$(v/u) = (y/h)(\partial h/\partial x) + F(y/h)\cdot(1/u)(\partial h/\partial t). \tag{20}$$

It is, therefore, seen that the ratio of (v/u) in the steady flow is linearly proportional to the surface gradient as Boussinesq assumed, while in unsteady flows his assumption is valid provided that $F(y/h) = (y/h)f(y/h)$.

Now, let determine the mathematical form of $F(y/h)$ and $f(y/h)$ when Boussinesq's assumption is valid. As $f(y/h) = dF(y/h)/d(y/h)$, so the following differential equation of $F(y/h)$ is obtained.

$$dF(y/h)/d(y/h) - F(y/h)/(y/h) = 0. \tag{21}$$

The solution of Eq.(21) is readily obtained and it is $F(y/h) = C\cdot(y/h)$, and thus $f(y/h) = C$. The definition of mean velocity yields that C is unity, so that the mathematical form of velocity distribution assumed by Boussinesq indicates uniform.

Next attention is directed to the magnitude of the curvature of stream line. Under the condition of steady flow, the local curvature is

$$(1/\rho) = (1/R_s)(y/h)\{1 + (dh/dx)^2\}^{3/2}/\{1 + (y/h)^2(dh/dx)^2\}^{3/2} \tag{22}$$

It is approximately

$$(1/\rho) \doteqdot (1/R_s)(y/h)\{1 + (3/2)(1 - y^2/h^2)(dh/dx)^2 + \ldots\}. \tag{23}$$

Fawer and Jaeger assumed the local curvature was expressible by

$$(1/\rho) \doteqdot (1/R_s)(y/h)^n, \tag{24}$$

and putting n = 1 leads the assumption of Fawer and Jaeger is equal to Eq.(23).

As the measurement of vertical velocity in the flow is practically impossible, so the verification of the foregoing analysis can not be directly made. The possible mean is to measure the pressure distribution by manometers and the verification can be made by the evaluation of pressure distribution influenced by the local vertical velocity.

(d) Velocity Distribution of Curvilinear Flows over Curved Boundary

Rapid changes in channel geometry and a discontinuity in bed slope produce the flow curvilinear and the most significant influence of such curvilinear flows is expressed by the centrifugal force. This subsection deals with the velocity distribution of curvilinear flows over curved boundary, for the purpose of further discussion of functional diversity of channel controls. As often seen in the previous description, the complete picture of velocity distribution of uniform flows in the light of the modern advances of theoretical hydrodynamics is not obtained, and therefore it is still difficult to express the velocity distribution of more complicated flow like the curvilinear flow in a mathematical form.

On the other hand, the curvilinear motion as a typical example of rapidly varied flow is a local phenomenon in the practical hydraulic engineering, so that the first approximate analysis of the velocity distribution based on the concept of irrotational motion will become available, and thus the flow will be assumed invicid and irrotational.

Taking the x-axis along the curved boundary, and the y-axis normal from the boundary, the irrotationality of flow yields the following expression of

$$(\partial v/\partial x) - (u/R) - (1 + y/R)(\partial u/\partial y) = 0, \quad (25)$$

in which, u and v: velocity components in both directions and $1/R$: local curvature of bed and a function of x.

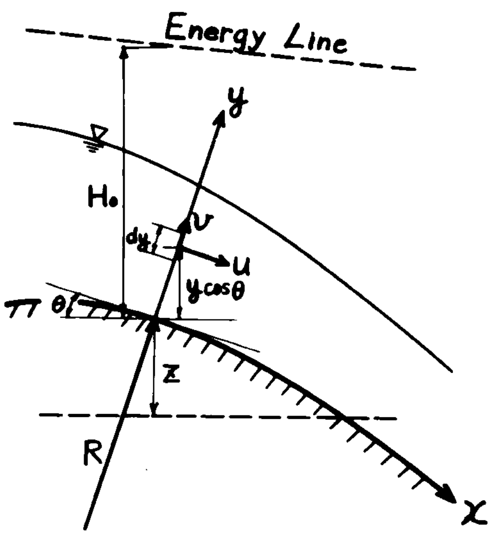

Fig. 1-4 Schematic diagram and notations of curved flow

In the further assumption that $(\partial v/\partial x)$ is extremely small compared with $(\partial u/\partial y)$, Eq.(25) becomes

$$(1 + y/R)(\partial u/\partial y) + (u/R) = 0. \quad (26)$$

Once integrating Eq.(26) with respect to y, the local component is, with the use of the constant discharge principle,

$$u = \{q/\log(1 + h/R)\}\{1/(R + y)\}, \quad (27)$$

and the surface and bottom velocities are also

$$u_s = u(R + y)/(R + h), \text{ and } u_b = u(R + y)/R. \quad (28)$$

The experimental verification of the foregoing approximation will be desirable, and Fig. 1-5 indicates the experimental data of velocity distribution over a round crested weir, which is equiped at the Hydraulics Laboratory, Kyoto University. Evidently, it is rather surprising that the foregoing assumption of irrotationality is available for the engineering analysis, and the velocity near the bed is still of large value. However, $(R + y)u$ is not constant, and decreases gradually near the bed as will be described.

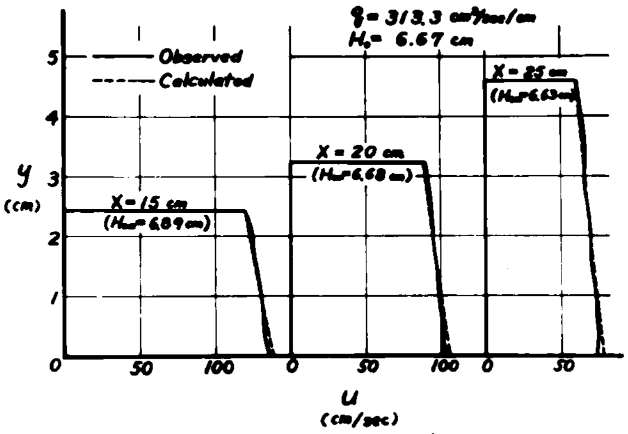

Fig. 1-5 Velocity profile of curved flow over round crested weir

The more complete picture of velocity distribution for the higher approximation must be obtained by the introduction of boundary layer theory, nevertheless the turbulent layer is so small that the character will not be verified by the present experimental instruments.

1 - 1 - 3 Pressure Distribution in Open Channel Flows

In common practice of hydraulic engineering for the evaluation of the physical characteristics of gradually and rapidly varied flows in open channels, the hydrostatic assumption of pressure even in a moving fluid is provided for the engineering approximation. The acceleration by the centrifugal action exerted in the curvilinear flow and the appreciable vertical acceleration of fluid itself yield the pressure distribution to be non-hydrostatic.

The knowledge of detail patterns of pressure distribution in a moving fluid is required for the development of hydraulics of open channel flows as classical science, and also closely associated with many practical problems encountered in hydraulic engineering. For example, a rapid change in fluid pressure at channel beds caused by a sudden increase or decrease of velocity over a spillway crest induces the negative pressure to a pronounced degree and the formation of cavities on the solid boundary.

In this section, the theoretical analysis and the experimental verification to the detail pattern of pressure distribution in a moving fluid required for the study of the physics of open channel flows will be concerned.

(a) Pressure Distribution in Gradually Varied Flows

Under the field of gravitational action, the pressure gradient with respect to the depth is provided by the acceleration of fluid itself, the component of gravity and the viscous or Reynolds stresses as seen in the foregoing. It is

$$-(1/\rho)(\partial p/\partial y) = g\cos\theta + (\text{ acceleration term }) + (\text{ friction term }). \tag{29}$$

Although this equation is strictly satisfied in actual open channel flows, the shearing effect to the vertical motion will be considered so small as to be practically ignored, so the vertical acceleration a_y in the accelerative flow is in a form of

$$a_y = -g\cdot\partial(p/\rho g + y\cos\theta)/\partial y. \tag{30}$$

The existence of acceleration at a point requires a proportionate decrease in the sum of potential and pressure head in the direction of upward. If an imaginary state in which the vertical acceleration becomes zero is assumed, the above quantity should be the same value of y throughout the zone of flow. Since this relationship is strictly satisfied only if the fluid is at rest, it is known as the principle of hydrostatics. However, such an effect of acceleration is commonly so small for gradually varied flows in channels where the channel geometry is not changed abruptly as to be negligible as a first approximation. The force by the flow pressure can then conveniently be evaluated by the assumption of hydrostatic pressure distribution.

$$p = \rho g \cos\theta (h - y). \tag{31}$$

This assumption prevails in the open channel flow of gradually varied regime, and nearly all solutions of problems in hydraulic engineering are derived by the hydrostatic law and the experimentation in laboratories confirms the assumption with a sufficient accuracy as the engineering practice.

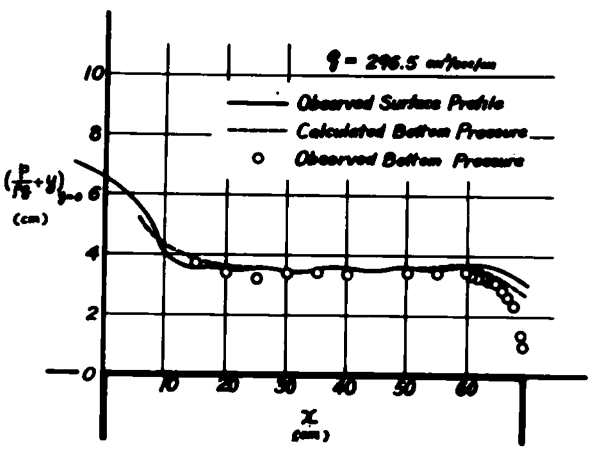

Fig. 1-6 Bed pressure distribution on broad crested weir

Figs. 1-6 and -7 indicate the magnitude of bed pressure of flows over a broad crested weir, which is 75 cm in length and 20 cm in width. It is seen that the appreciable hydrostatic pressure distribution prevails in the middle part of weir. On the other hand, the non-hydrostatic influences produced by the curvilinear motion are extremely large in the inlet and free overfall sections.

(b) Pressure Distribution in Flows with Appreciable Vertical Acceleration

The previous assumption of hydrostatic distribution in gradually varied flows is only convenient for the practical calculation in hydraulic engineering as a first approximation, so that the detail pattern of pressure related to the dynamical relationship for actual fluid requires the evaluation of vertical acceleration in a moving fluid, which has been investigated by Boussinesq, Fawer, Sérre and the author.

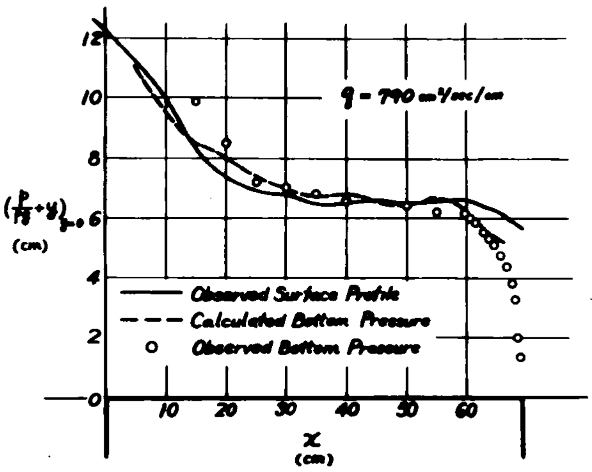

Fig. 1-7 Bed pressure distribution on broad crested weir

The deduction of Boussinesq in the pressure distribution of flows with appreciable curvature of free surface is following. From Eq.(29), the pressure gradient is

$$-(1/\rho)(\partial p/\partial y) = g\cos\theta + (\partial v/\partial t) + u(\partial v/\partial x) + v(\partial v/\partial y)$$
$$= g\cos\theta + (\partial v/\partial t) + u^2\partial(v/u)/\partial x. \qquad (32)$$

The evaluation of pressure in a moving fluid requires the knowledge of velocity profiles. As seen in Eq.(16) of 1-1-2-(c), Boussinesq used the following assumptions.

(1) $(v/u) = (y/h)\{(\partial h/\partial x) + (1/u_m)(\partial h/\partial t)\}$,

(2) $u \fallingdotseq u_m$,

(3) All higher terms depending on squares and products of derivatives are negligible.

Integrating Eq.(32) from y to h under the above assumptions, the pressure distribution becomes in a form of

$$(p/\rho g) = (h - y)\cdot\cos\theta + (u_m^2/2g)(h^2 - y^2)/h\cdot\{(\partial^2 h/\partial x^2) + (2/u_m)(\partial^2 h/\partial x\partial t) + (1/u_m^2)(\partial^2 h/\partial t^2)\}. \qquad (33)$$

Eq.(33) indicates the vertical acceleration in a moving fluid leads the pressure distribution to be non-hydrostatic and the magnitude of curvature of stream line also to be appreciable. It is well known this is a basic relationship in the momentum approach of open channel flows developed by Boussinesq.

In 1937, Fawer derived the second approximation of open channel flows characterized by the non-hydrostatic pressure by means of the energy approach of analysis and classified the surface profiles of flow. Although Boussinesq assumed the curvature of stream line increased linearly from the bed to the free surface, the assumption of Fawer on the curvature in his treatment was in a form of

$$1/(R + y) = (1/R_s)(y/h)^n, \tag{34}$$

where $1/R_s$: curvature at the free surface. Putting $u \doteqdot u_m$, the pressure distribution is obtained, as

$$\begin{aligned} p/\rho g &= (h - y)\cos\theta + \int_y^h \{u^2/g(R + y)\}dy \\ &= (h - y)\cos\theta + u_m^2/gR_s(1 + n)\cdot\{1 - (y/h)^{n+1}\}. \end{aligned} \tag{35}$$

Theoretical deductions of Boussinesq and Fawer are obtained under the assumption of constant velocity profiles. As seen in the foregoing, the magnitude of velocity in the x-direction, however, changes with the depth of water, so that these results of higher orders are still remained approximate.

The author[7)] obtained the pressure distribution of flows with appreciable vertical acceleration in more definite form and it is as follows. From Eq.(20) in 1-1-2-(c), the ratio of v to u is

$$v/u = (y/h)(\partial h/\partial x) + (1/u)\cdot F(y/h)\cdot(\partial h/\partial t),$$

and the evaluation of pressure distribution with respect to the depth is thus calculated, by the use of Eq.(32).

$$p/\rho g = (h - y)\cos\theta + (hu_m^2/g)(\partial^2 h/\partial x^2)\int_m^1 mf^2(m)dm + (hu_m/g)(\partial^2 h/\partial x\partial t)$$
$$\int_m^1\{mf(m) + f(m)F(m)\}dm + (h/g)(\partial^2 h/\partial t^2)\int_m^1 F(m)dm + (h/g)(\partial u_m/\partial t)(\partial h/$$
$$\partial x)\int_m^1 mf(m)dm + (1/g)(\partial h/\partial t)^2\int_m^1\{f(m)F(m) - mf(m)\}dm + (u_m/g)(\partial h/\partial x)$$

$$(\partial h/\partial t)\int_m^1\{f(m)F(m) + mf'(m)F(m) - mf(m) - m^2f'(m) - mf^2(m)\}\,dm - (u_m^2/g)(\partial h/\partial x)^2\int_m^1 mf^2(m)\cdot dm, \qquad (36)$$

in which m: y/h and f'(m): df/dm.

Denoting the pressure deviation from the hydrostatic pressure by Δp as Jaeger did, the non-hydrostatic term of pressure distribution in the above expression is

$$\Delta p/\rho g = (hu_m^2/g)(\partial^2 h/\partial x^2)\int_m^1 mf^2(m)dm + (hu_m/g)(\partial^2 h/\partial x\partial t)\int_m^1\{mf(m) + f(m)F(m)\}dm + (h/g)(\partial^2 h/\partial t^2)\int_m^1 F(m)dm + (h/g)(\partial u_m/\partial t)(\partial h/\partial x)\int_m^1 mf(m)dm + (1/g)(\partial h/\partial t)^2\int_m^1\{f(m)F(m) - mf(m)\}dm + (u_m/g)(\partial h/\partial x)(\partial h/\partial t)\int_m^1\{f(m)F(m) + mf'(m)F(m) - mf(m) - m^2f'(m) - mf^2(m)\}dm - (u_m^2/g)(\partial h/\partial x)^2\int_m^1 mf^2(m)dm.$$

This equation is derived without any negligence of higher terms depending on squares and products of time and spatial derivatives, and it can be readily reduced to the equation of Boussinesq, if higher terms are ignored and the velocity is assumed uniform. For steady flows, $\Delta p/\rho g$ becomes

$$\Delta p/\rho g = (q^2/g)\int_m^1 mf^2(m)dm\{(1/h)(d^2h/dx^2) - (1/h^2)(dh/dx)^2\}, \qquad (38)$$

The dotted lines in Figs. 1-6 and -7 indicate the calculated curves of bed pressure considered by the non-hydrostatic influence resulted from the acceleration of vertical motion and the theoretical analysis is closely agreed with the measured pressure. As the flow over a broad crested weir is commonly near critical, so it is seen that the influence of vertical acceleration is not ignored.

(c) Pressure Distribution of Curved Flow

The pressure distribution of curvilinear flow is provided by the acceleration of gravity and the centrifugal action, and the vertical acceleration of fluid flow itself is considered small compared with the foregoing two actions if the coordinate system is chosen along the solid boundary as seen in the previous section. The equation of motion in the upward direction is

$$(\partial v/\partial t) + \{Ru/(R + y)\}(\partial v/\partial x) + v(\partial v/\partial y) - u^2/(R + y) = -g\cos\theta - (1/\rho)(\partial p/\partial y). \tag{39}$$

When the above assumption will be used for the approximate behaviours of pressure distribution of curvilinear motion, Eq.(39) becomes

$$-(1/\rho)(\partial p/\partial y) = g\cos\theta - u^2/(R + y). \tag{40}$$

Inserting the velocity distribution of u in Eq.(27) into the above equation, the pressure distribution is obtained by integrating from y to the free surface and it is

$$p/\rho g = (h - y)\cos\theta + (u_b^2R^2/2g)\{1/(R + h)^2 - 1/(R + y)^2\}. \tag{41}$$

The first term indicates evidently the hydrostatic pressure and the second the influence of centrifugal force due to the curvilinear motion.

The bed pressure p_b, which is one of the most significant value for engineering problems encountered in design of overflow spillway, is then

$$p_b/\rho g = h\cos\theta\{1 - (u_b^2/2g\cos\theta)(2R + h)/(R + h)^2\}. \tag{42}$$

At the end of free overfall, the pressure becomes atmospheric and the product of **brinking** depth h_b and local curvature is

$$h_b/R_b = 2(gR_b\cos\theta/u_s^2 - 1), \tag{43}$$

in which $1/R_b$: local curvature at the free overfall.

The deviation of pressure from the hydrostatic pressure is also given by

$$\Delta p/\rho g = (u_b^2R^2/2g)\{1/(R + h)^2 - 1/(R + y)^2\}. \tag{44}$$

Fig. 1-8 indicates the measured bed pressure of flows over a round crested weir of 30 cm in diameter. The close agreement between the theoretical results and the measured pressure will be seen in the downstream side from the crest. In the figure, X denotes the distance from the downstream edge.

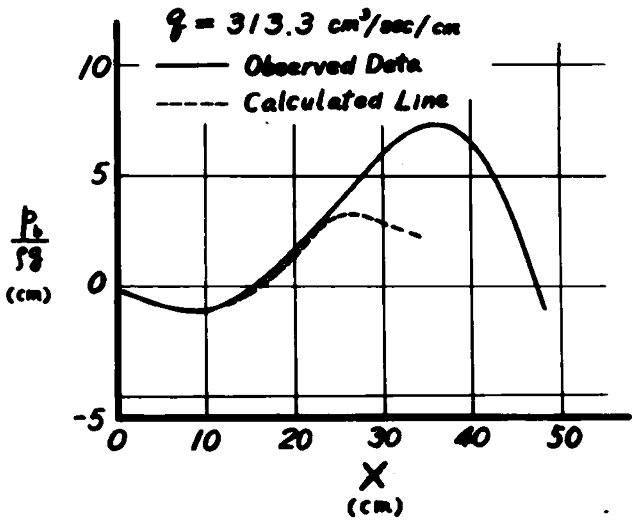

Fig. 1-8 Bed pressure over round crested weir

(d) Correction Parameter of Pressure in Terms of Jaeger's Notation

The previous subsection deals with the detail patterns of pressure distributions in straight stream lined flows and curvilinear flows. In the common procedure of hydraulic analysis of open channel flows in uniform channels, mean values of total head H_0 above the channel bed and of momentum flux M_0 over a whole section of flow, as will be introduced in the next section, are widely used, and the magnitude of pressure also will be evaluated as a mean value of local pressure. For this reason, Jaeger[13)] introduced the correction parameter of pressure like Coriolis and Boussinesq parameters of velocity as follows.

$$\lambda = (1/Qh\cos\theta)\int(p/\rho g + y\cos\theta)u\,dA, \qquad (45)$$

for energy approach of one dimensional analysis and

$$\lambda' = (1/A\bar{y}_G\cos\theta)\int(p/\rho g)\,dA, \qquad (46)$$

for momentum approach, in which y_G is the distance from the free surface to the centroid of flow area, though the original symbols in his paper are different from Eqs.(45) and (46).

Almost hydraulic problems encountered in the usual engineering purpose of gradually varied flows involve the hydrostatic law of pressure distribution as described in Eq.(31). Inserting Eq.(31) into Eqs.(45) and (46)

$$\lambda = (1/Qh\cos\theta)\int \cos\theta(h - y + y)u\,dA = 1, \tag{47}$$

and

$$\lambda' = (1/Ay_G\cos\theta)\int \cos\theta(h - y)\,dA = 1. \tag{48}$$

The pressure distribution in curvilinear flows and accelerative flows are extremely influenced by the centrifugal force and the vertical acceleration, as described in the foregoing section, so that the local pressure at a point will be expressed as the sum of hydrostatic pressure and non-hydrostatic influence and it is

$$p/\rho g = (h - y)\cos\theta + (\Delta p/\rho g). \tag{49}$$

Putting Eq.(49) into Eqs.(45) and (46)

$$\lambda = 1 + (1/Qh\cos\theta)\int(\Delta p/\rho g)u\,dA, \tag{50}$$

and

$$\lambda' = 1 + (1/Ay_G\cos\theta)\int(\Delta p/\rho g)\,dA. \tag{51}$$

Evidently, when all stream lines are concave downwards, $\Delta p < 0$ at every point, and thus λ and λ' become smaller than unity, and concave stream lines in the upward direction indicate that λ and λ' are larger than unity.

When the flow is extremely accelerative in the y-direction, the non-hydrostatic pressure due to the vertical acceleration is indicated by Eq.(37), and therefore, λ and λ' in terms of the two dimensional cases are expressible in forms of

$$\begin{aligned}\lambda = 1 &+ (1/g\cos\theta)\Big[u_m^2(\partial^2 h/\partial x^2)\int_0^1 f(m)\int_m^1 mf^2(m)(dm)^2 + u_m(\partial^2 h/\partial x\partial t)\int_0^1 f(m)\\ &\int_m^1\{mf(m) + f(m)F(m)\}(dm)^2 + (\partial^2 h/\partial t^2)\int_0^1 f(m)\int_m^1 F(m)(dm)^2 + (\partial u_m/\partial t)\\ &(\partial h/\partial x)\int_0^1 f(m)\int_m^1 mf(m)(dm)^2 + (1/h)(\partial h/\partial t)^2\int_0^1 f(m)\int_m^1\{f(m)F(m) - mf(m)\}\\ &(dm)^2 + (u_m/h)(\partial h/\partial x)\cdot(\partial h/\partial t)\int_0^1 f(m)\int_m^1\{f(m)F(m) + mf'(m)F(m)\end{aligned}$$

$$- mf(m) - m^2 f'(m) - mf^2(m)\}(dm)^2 - (u_m^2/h)(\partial h/\partial x)^2 \int_0^1 f(m) \int_m^1 mf^2(m) (dm)^2 \Big] , \quad (52)$$

and

$$\lambda' = 1 + (2/g\cos\theta)\Big[u_m^2(\partial^2 h/\partial x^2)\int_0^1\int_m^1 mf^2(m)(dm)^2 + u_m(\partial^2 h/\partial x\partial t)\int_0^1\int_m^1\{mf(m) + f(m)F(m)\}(dm)^2 + (\partial^2 h/\partial t^2)\int_0^1\int_m^1 F(m)(dm)^2 + (\partial u_m/\partial t)(\partial h/\partial x)\int_0^1\int_m^1 mf(m) (dm)^2 + (1/h)(\partial h/\partial t)^2\int_0^1\int_m^1\{f(m)F(m) - mf(m)\}(dm)^2 + (u_m/h)(\partial h/\partial x) (\partial h/\partial t)\int_0^1\int_m^1\{f(m)F(m) + mf'(m)F(m) - mf(m) - m^2 f'(m) - mf^2(m)\}(dm)^2 - (u_m^2/h)(\partial h/\partial x)^2\int_0^1\int_m^1 mf^2(m)(dm)^2 \Big] . \quad (53)$$

Furthermore, if the velocity is uniform in the steady flow, $f(m) = 1$ and then

$$\lambda = 1 + (q^2/3gh^2\cos\theta)\{(d^2h/dx^2) - (1/h)(dh/dx)^2\}, \quad (54)$$

and

$$\lambda' = 1 + (2q^2/3gh^2\cos\theta)\{(d^2h/dx^2) - (1/h)(dh/dx)^2\} , \quad (55)$$

thus, the non-hydrostatic influence in λ' is twice larger than that of λ , and it is readily understood that both parameters indicate the influence of mean curvature of stream line due to the vertical acceleration.

For the curvilinear motion, the deviation of pressure head from the hydrostatic pressure head is

$$\lambda = 1 + (u_s^2/2gh\cos\theta)\Big[1 - \{1/2\log(1 + h/R)\}\cdot(2Rh + h^2)/R^2\Big] \quad (56)$$

or simply for $R \gg h$, it is

$$\lambda \doteqdot 1 - (u_s^2/2gR\cos\theta)(1 + h/4R + \cdots), \quad (57)$$

and

$$\lambda' = 1 - u_s^2/gR\cos\theta . \quad (58)$$

These equations indicate the influences of mean curvature of stream line due to the centrifugal action in curvilinear motion.

1 - 1 - 4 One Dimensional Approaches of Analysis in Open Channel Flows

Of evidence is that the physical behaviours of open channel flows are the solution of fundamental hydrodynamic equations based on the Newtonian principles of motion. Main attention in the past two sections has been accented on means of evaluating the distributions of velocity and pressure throughout the whole zone of flow, to permit the solution of problems which involve specific details of the accelerative patterns. In many cases, however, the knowledge of such details is unnecessary to the bulk solution of the problem, or else the detailed solution is so complex that a gross approximation is all that can be expected. This is accomplished presuming the zone of flow under consideration to consist a single stream tube characterized at a section by a mean velocity of flow, a pressure and a mean elevation. Variation of flow characteristic across a section is thus ignored, and heed is given only to the change of mean values in the direction of flow. The result is the one dimensional approach of analysis, which formed the basis of the 19 th century hydraulics, and is followed by the characteristics of confined flow.

Many scientists and engineers in the 19 th century solved their problems in the flow of open channels with the use of the one dimensional equation of energy approach based on the Bernoulli's equation under the assumption of hydrostatic pressure distribution. This method was refined by Fawer[4)] in 1937, who extended the first approximation of energy equation to the second one characterized by the non-hydrostatic pressure distribution. On the other hand, the one dimensional approach of momentum was initiated by Boussinesq, and the first and second approximations derived by him are widely seen in many hydraulic literature.

G.H. Keulegan and G.W. Patterson[2)], in 1943, succeeded the most rigorous deduction of the one dimensional equation of open channel flows for a uniform channel from the original equation of Reynolds for turbulent flows. Highly simple and elaborate methods of momentum and energy in the one dimensional analysis were completed by Iwagaki[12] in 1955. His momentum equation for the hydrostatic flow is the same expression as that derived by Keulegan and Patterson. He also applied these methods to the flow with lateral in and outflow and refined the treatments of H. Favre[22] and de Marchi[23]. F. Sérre[5] and the author[7] made the formulation of one dimensional equations in a uniform channels by means of both approaches of energy and momentum and also their treatments to the non-hydrostatic flows.

Recently, K.O. Friedlich[24] studied the mathematical characters of the Eulerian equation of motion with the aid of the purturbation method, and concluded that the first approximation in the shallow water wave theory is similar to the equation of open channel flows and the second approximation is characterized by a cnoidal wave[25], which is also the basis of the second approximation of one dimensional procedure as seen in the later section.

In any case of confined and unconfined flows, the dynamics of open channel flows is derived by the hydrodynamic principles. The solution of problems in hydraulics, therefore, obtained by the Navier-Stokes or Reynolds equations under given boundary and initial conditions associated with the problem under consideration indicates the hydraulic behaviours of flow. However, until the present day no general methods have become available for the integration of the original equation owing to the great mathematical difficulty.

The one dimensional approach becomes thus effective as an approximate equation in the physics of open channel flows. The evaluation of the magnitude of errors, however, which can result from the method of approximation, is of essential importance to express the motion of fluid flows in terms of the one dimensional method of analysis. The surface resistance and the resulting velocity profile from the momentum interchange of turbulence yields the irregular distribution of energy and momentum. The influence of such a dispersion is characterized by correction coefficients of Coriolis, Boussinesq and Jaeger for mean flows. For the practical calculation in design problems, the uniform distribution in velocity and the hydrostatic pressure are commonly assumed and the usual coefficients of Coriolis and Jaeger are unity. The unequivalency between the momentum and energy approaches is resulted from the practical assumption, ignored the exact physical behaviours of flows.

A large number of experiments demonstrate that the pressure distribution in a moving fluid is assumed hydrostatic as practical approximation. In zones of separation near positive and negative steps and in the flow near estuaries, however, correction coefficients become surprisingly large. Another important case is the coefficient of Coriolis is of essential character in itself for the physics of open channel flows like the Vedernikov criterion for formation of hydraulic instability, which is substantially different from the hydrodynamic stability. The fact must be borne in mind that the valid evaluation of coefficients in the basic equation of one dimensional analysis can represent the true feature in the physics of open channel flows and considerable errors may be involved in some particular cases.

(a) Application of Energy Theorem to One Dimensional Approach

Common practice to deduce the one dimensional equation of open channel flows in the past has been to apply the energy theorem derived by the Newtonian principles of motion to the flow. Although the recourse of procedure is unnecessary to express, a brief conclusive deduction of the one dimensional energy equation will be described for the further discussion of the present study.

Taking the coordinate system at the channel bed, as seen in Fig. 1-4, the total energy past the section 1 is

$$\int_1 \{(u^2 + v^2)/2g + p/\rho g + y\cos\theta + z\} u dA, \tag{59}$$

and that past the section 2 is

$$\int_2 \{(u^2 + v^2)/2g + p/\rho g + y\cos\theta + z\} u dA \tag{60}$$

The difference of energy between two sections is equal to the increase of energy bounded by two sections and the loss of energy by the boundary resistance. The energy theorem thus leads to the following equation of

$$\int_2 \{(u^2 + v^2)/2g + p/\rho g + y\cos\theta + z\} u dA - \int_1 \{(u^2 + v^2)/2g + p/\rho g + y\cos\theta + z\} u dA + (\partial/\partial t)\int\{(u^2 + v^2)/2g + z + y\cos\theta\}(1 + y/R) dA dx + \int(\tau/\rho g)(1 + y/R) u_b ds dx = 0, \tag{61}$$

and thus the final expression is

$$(\partial/\partial x)\int\{(u^2 + v^2)/2g + p/\rho g + y\cos\theta + z\} u dA + (\partial/\partial t)\int\{(u^2 + v^2)/2g + y\cos\theta + z\}(1 + y/R) dA + \int(\tau/\rho g)(1 + y/R) u_b ds = 0. \tag{62}$$

Under the common state of open channel flows, the magnitude of vertical velocity is small compared with the velocity of running water, and $R \gg h$, thus Eq.(62) becomes

$$(\partial/\partial x)\int(u^2/2g + p/\rho g + y\cos\theta + z) u dA + (\partial/\partial t)\int(u^2/2g + y\cos\theta + z) dA + (\tau s/\rho g) u_b = 0. \tag{63}$$

Eq.(63) is a different form of the common energy equation derived by many investigators, and readily transformed into the usual equation of unsteady open channel flows, under the hydrostatic assumption in pressure, which is

$$\beta(\partial u_m/\partial t) + \alpha u_m(\partial u_m/\partial x) + g\cos\theta(\partial h/\partial x) + g(\beta - \alpha)(u_m/2g)(\partial A/\partial t)$$
$$= g\sin\theta - (g\tau/\rho gR)(u_b/u_m). \qquad (64)$$

The above equation is the same expression of Iwagaki[2], and for more general cases, in which the flow will be curvilinear or influenced by the appreciable vertical velocity and acceleration, Eqs.(62) and (63) expressed in terms of the foregoing results will be valid.

Now, the specific energy per unit volume of fluid over a bed H_o will be introduced. The parameter is extremely available for the analysis of steady flow problems in hydraulic engineering as frequently observed in the later section. From the definition,

$$H_o = (1/Q)\int\{(u^2 + v^2)/2g + p/\rho g + y\cos\theta\}u\,dA, \qquad (65)$$

so that H_o for two dimensional flows of appreciable vertical acceleration, which is one of basic quantity for the analysis of unsteady hydraulic problems like translation wave of finite amplitude, is as follows.

$$\begin{aligned} H_o = {} & (u_m^2/2g)\int_0^1 f^3(m)dm + h\cos\theta + (hu_m^2/g)(\partial^2 h/\partial x^2)\int_0^1 f(m)\int_m^1 mf^2(m)(dm)^2 \\ & + (hu_m/g)(\partial^2 h/\partial x\partial t)\int_0^1 f(m)\int_m^1\{mf(m) + f(m)F(m)\}(dm)^2 + (h/g)(\partial^2 h/\partial t^2) \\ & \int_0^1 f(m)\int F(m)(dm)^2 + (u_m/g)(\partial h/\partial x)(\partial h/\partial t)\Big[\int_0^1 f(m)\int_m^1\{f(m)F(m) + mf'(m) \\ & \cdot F(m) - mf(m) - m^2 f'(m) - mf^2(m)\}(dm)^2 + \int_0^1 mf^2(m)F(m)dm\Big] + (h/ \\ & g)(\partial u_m/\partial t)(\partial h/\partial x)\int_0^1 f(m)\int_m^1 mf(m)(dm)^2 + (u_m^2/2g)(\partial h/\partial x)^2\Big\{\int_0^1 m^2 f^3(m)dm \\ & + 2\int_0^1 f(m)\int_m^1 mf^2(m)(dm)^2\Big\} + (1/2g)(\partial h/\partial t)^2\Big\{\int_0^1 f(m)F^2(m)dm + 2\int_0^1 f(m)\int_m^1 \\ & \{f(m)F(m) - mf(m)\}(dm)^2. \end{aligned} \qquad (66)$$

Under the flow condition of small vertical velocity, as usual gradually varied flows, the approximate expression of H_o becomes

$$H_o = (1/Q)\int(u^2/2g + p/\rho g + y\cos\theta)u\,dA, \qquad (67)$$

so that, with the use of the one dimensional procedure,

$$H_o = (\alpha Q^2/2gA^2) + \lambda h\cos\theta. \qquad (68)$$

The characteristics of H_0 in Eq.(68) will be described in the later section, and the energy equation of one dimensional approach of steady flows is then

$$(dH_0/dx) = \sin\theta - (\tau/\rho gR)(u_b/u_m). \qquad (69)$$

The steady flow problems for constant discharge encountered in the trace of surface profiles of water and the like for design of conveyance and control structures will be solved by two equations of (68) and (69).

(b) Momentum Theorem of One Dimensional Approach

From the most important second law of Newtonian principles of motion, the momentum theorem is deduced. The application of this theorem to the flow was initiated by Boussinesq and followed by many investigators. This subsection deals with a brief summary of the momentum equation for the further research.

The increase of momentum between two sections is equal to the momentum change in the fluid enclosed by both sections, the pressure difference, the gravity force acting on the total mass and the shear along the solid boundary, and it follows

$$\int_2 \rho u^2 dA - \int_1 \rho u^2 dA + (\partial/\partial t)\int \rho u(1 + y/R)dAdx = \int \rho g \sin\theta (1 + y/R) dAdx - \tau s dx - \int (\partial p/\partial x)dAdx$$

and thus

$$(\partial/\partial t)\int \rho u(1 + y/R)dA + (\partial/\partial x)\int \rho u^2 dA + \int (\partial p/\partial x)dA = \rho g \sin\theta \int (1 + y/R)dA - \tau s . \qquad (70)$$

When the channel bottom is gradually varied, and $R \gg h$, Eq.(70) becomes

$$(\partial/\partial t)\int (u/g)dA + (\partial/\partial x)\int (u^2/g)dA + \int (1/\rho g)(\partial p/\partial x)dA = A\sin\theta - (\tau/\rho g)s. \qquad (71)$$

This equation is the basic equation for open channel flows in straight and gradually varied slopes, and the usual form like Keulegan's is deduced under the hydrostatic condition in pressure distribution.

If the channel geometry is uniform, $(\partial A/\partial x) = b(\partial h/\partial x)$ and the bed is very mild, Eq.(71) is transformed into

$$(\partial/\partial t)\int(u/g)dA + (\partial/\partial x)\int(u^2/g + p/\rho g)dA = A\sin\theta - (\tau/\rho g)s \tag{72}$$

In the foregoing subsection, the concept of mean specific energy was introduced, and here, the concept of momentum flux M_o, which is defined as the sum of the momentum flux past a section and the pressure at that section divided by the specific weight, will be again introduced, so that Eq.(72) becomes

$$(\partial/\partial t)\int(u/g)dA + (\partial M_o/\partial x) = A\sin\theta - (\tau/\rho g)s, \tag{73}$$

and

$$M_o = \int(u^2/g + p/\rho g)dA. \tag{74}$$

With the use of the one dimensional expression, M_o becomes

$$M_o = (\rho Q^2/gA) + \lambda' A y_G \cos\theta \quad . \tag{75}$$

The characteristics of M_o will be treated in the later section, and the momentum equation of one dimensional approach of steady flows in uniform channels of mild slope is then,

$$(1/A)(dM_o/dx) = \sin\theta - (\tau/\rho gR). \tag{76}$$

1 - 1 - 5 Basic Principles of Boundary Layer Growth in Open Channel Flows

The fluid flow which is carried from a reservoir to a chute is retarded by the surface resistance along the solid boundary and the velocity gradient is quite large within a thin layer near the boundary, if the distance the fluid has travelled is not so long. This zone of retarded fluid is characterized as a boundary layer which increases continually to the free surface. It is a matter of common observation that the boundary layer exists even in the flow of open channels near the inlet of chutes, while the analysis of the physics of such transitional behaviours in flow, hitherto, was made in common by the usual method of one dimensional approach.

Quite different characters of boundary layer growth in the flow of open channel from those of flow near an obstacle or around an airfoil are imposed by the free surface, which is sensible for other influences by channel geometry and boundary. The open channel flow is then called as a confined flow. Considerations on the interdependency of the physics in the boundary layer to that in the main flow make useful solutions of problems on the boundary layer growth for practical application in hydraulics, while the behaviour of unconfined flow can be practically calculated without almost any modification of its original equation.

The history of investigations of boundary layer growth in open channel flows was initially related to the problem of air-entrainment process into the fluid over a steep chute suggested by E.A. Lane[26]. Many hydraulic engineers, thereafter studied the process associated with the practical design in spillways and chutes and almost verified with field observations without establishing the dynamics of air-entrainment. Nearly all procedures[27] have ever studied, however, are only related to the empirical relationship based on the well known equation of Prandtl and Schlichting in the turbulent boundary layer growth on a flat plate. In 1952, first investigation of the interdependency between both flows of boundary layer and main flow was published by G. Halbronn[28] and in the same year, A. Craya and J.W. Delleur[8] also studied this subject associated with the problem in horizontal and sloping channels, with results of refined interpretation of the behaviours in a flow near critical regime. More recently, W.J. Bauer[29] conducted the systematic experiments of velocity profiles of turbulent layer of accelerative flows, and Delleur[30] attempted to apply the problem to the formulation of hydraulic performance of control structures like a broad crested weir and long weir. The author[31] also studied the theoretical basis of hydraulics in boundary layer growth and conducted a large number of experiments and obtained some important conclusion of initially supercritical flows in 1957 and discussed the general characters of boundary layer growth in open channel flows and its related problems in 1958[32] and 1959[33].

In this subsection, the purpose of the study is to reveal the general character of turbulent layer, which is of more importance in the engineering problem, and its basic relationship under steady regime, while the final stage of establishment of this problem stands still far away owing to the great complicated phenomena combined with turbulence and non-linear mechanical characters in original flows.

(a) Basic Equations of Motion by Means of Boundary Layer Theory

The introduction of boundary layer into the open channel flow near the inlet of channels and the like leads the open channel flow into two zones of the boundary layer, which is characterized by the influence of appreciable boundary resistance, and of main flow, in which the potential character becomes predominant. The physical principles of open

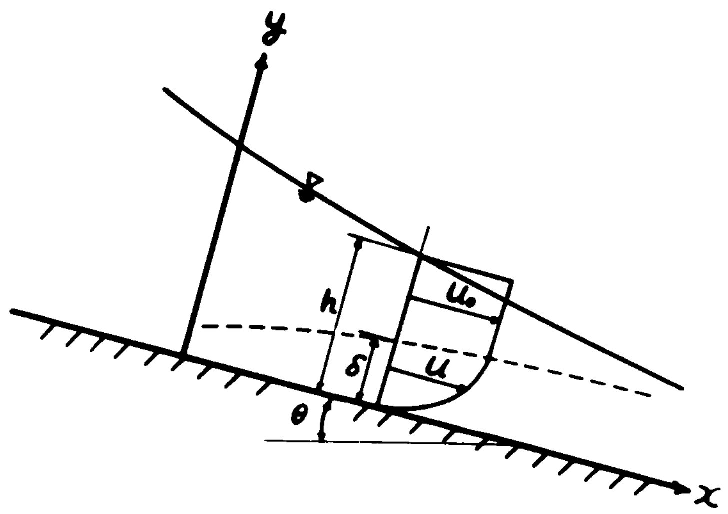

Fig. 1-9 Schematic representation of boundary layer growth in open channel flows

channel flows thus are described by combining the dynamical expressions of two zones of flows through the mass conservation law.

The equation of motion in the main flow above the virtual boundary located at the boundary layer thickness will be derived.

$$(u_0^2/2g) + h\cos\theta = \bar{H}_0 + x\sin\theta \ , \qquad (77)$$

where u_0: velocity in the main flow and $\bar{H}_0$: total head at the origin, if the velocity profile is of uniform type.

The boundary layer equation is, under the assumption of hydrostatic law,

$$(d/dx)\int_0^\delta u^2 dy - u_0(d/dx)\int_0^\delta u dy = g\sin\theta\int_0^\delta (1 + y/R)dy - g\cos\theta\,(dh/dx)\cdot\delta - (\tau/\rho). \qquad (78)$$

If the local curvature is very small compared with other variables or the channel is straight, Eq.(78) becomes familiar to hydraulic engineers and it is

$$(\tau/\rho) = g\delta\{\sin\theta - \cos\theta(dh/dx)\} + (d/dx)\int_0^\delta u(u_0 - u)dy - (du_0/dx)\int_0^\delta u dy, \qquad (79)$$

or inserting the Bernoulli equation into Eq.(79) and introducing the displacement and momentum thicknesses defined by

$$u_0\delta_* = \int_0^\delta (u_0 - u)dy, \quad \text{and} \quad u_0^2\vartheta = \int_0^\delta u(u_0 - u)\cdot dy, \qquad (80)$$

the boundary layer equation becomes

$$(\tau/\rho u_0^2) = (C_f/2) = (d\vartheta/dx) + (1/u_0)(du_0/dx)(2\vartheta + \delta_*). \qquad (81)$$

Designating the ratio of (δ_*/ϑ) by H, Eq.(81) is again

$$(HC_f/2) = (d\delta_*/dx) + \{(2 + H)/u_0\}\cdot(du_0/dx)\delta_* - (1/H)(dH/dx)\delta_*. \qquad (82)$$

Two equations in the main flow and boundary layer are connected together with the constant discharge principle of

$$q = u_0(h - \delta_*). \qquad (83)$$

The boundary layer growth encountered in engineering problems of calculation of the skin friction and the like is described as a combined solution of three equations of (77), (82) and (83). Before the calculation procedure, however, is made, the knowledge of velocity profile across a section must be required. Despite of the introduction of boundary layer concept into the open channel flow, the usual hydraulic equation of Bresse for surface profiles is obtained[31)] if the displacement thickness is eliminated from Eq.(81) with the use of Eq.(77).

(b) Velocity Distribution in Turbulent Layer

Although many investigation have been made to this subject, a rational theory for fully developed turbulent flow is still non-existent and in view of the extreme complexity of such flows it will remain so for a considerable time. Only the way possible to approach the fruitful success in this problem is to derive the solution or formulas of steady uniform flows. For many years, thus, engineers and scientists have enforced to establish the velocity distribution in pipes, conduits and open channels under such flow characteristics. And the analysis of the boundary layer growth by the application of momentum equation to assumed families of velocity profile remains popular, since engineers have to make computations relating to the turbulent skin friction and the like with the aid of some developments of empirical methods.

In this subsection, a brief conclusion of velocity profile and related skin friction of turbulent layer obtained from the experiments conducted at the Hydraulics Laboratory, Kyoto University, is concerned. The flume is of length of 11 m and its slope is variable from 0 to 30 degree. The side wall and bed consist of a very smooth lucite and the Manning roughness is estimated 0.0084 (m-sec) in average. At the entrance to channel there is a reach where the hypothesis of flow without the curvature is not valid and which makes the velocity near the bottom faster than that in the upper flow. A lucite guide vane, therefore, was set up to make the velocity at the reach uniform.

The problem of laminar layer is largely mathematical, since it is certain that the fundamental mechanical principles are fully understood and that the equations adequately describe the phenomena when the flow may be regarded as imcompressible. T. Ishihara, Y. Iwagaki and T. Goda[34] proved it was parabolic as a close approximation, though the laminar layer has of less importance in engineering problems.

In fully developed turbulent layer, which is mainly subjected to study, Reynolds has refined the method with insufficient results to make the theoretical analysis possible. A somewhat different method of simplifying the Navier-Stokes relation was proposed by Boussinesq. However, great progress of the problem was achieved with the introduction of mixing length theory by Prandtl, which clearly formulated the actual interrelationship of the Reynolds and Boussinesq parameters and in the same period Taylor and von Kármán developed the theory, clothed it in more elegant mathematical form. More recently, F.H. Clauser[18] divided the velocity distribution into two parts of inner and outer. On the other hand, the effort in establishment of the power law still continues from the practical point of view of dimensional analysis in classical hydraulics.

In the past the popular form has been of the power type $u_0 \propto y^{(1/7)}$ which is known as the Blasius 7 th law, i.e.

$$C_f = (2\tau/\rho u_0^2) = 0.0450 R_\delta^{-(1/4)}, \tag{84}$$

and

$$(u/u_0) = (y/\delta)^{1/7}. \tag{85}$$

W.J. Bauer[27] studied the velocity profile of turbulent layer in open channel flows with systematic experiments and concluded the shape of velocity profile associated with the development of turbulent layer on steep slopes was better approximated in most researches by an expression of the power type than that of the logarithmic form and it is valid at large values of Reynolds number. Fig. 1-10 indicates the relationship between C_f and the Reynolds number for δ_*. It is seen that C_f is slightly less than those of Blasius and Bauer within the whole range of Reynolds numbers. The prediction, however, of validity of the 7 th law will be expected.

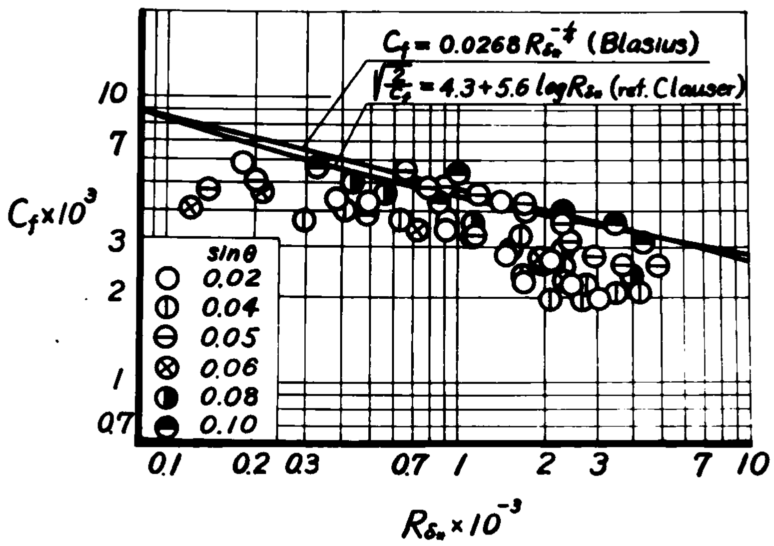

Fig. 1-10 Relationship between C_f and Reynolds number for displacement thickness

As C_f is assumed proportional to the m th power of Reynolds number in the power law, so between m and H, it is found that

$$m(1 + H) + 2(H - 1) = 0, \tag{86}$$

from the dimensional analysis. It indicates H is a constant of (9/7) without the range of Reynolds number, if the velocity profile is of Blasius type. Fig. 1-11 describes the growth of H with the increase of Reynolds number for displacement thickness. Evidently, H is not constant and increases to a constant value determined by given flow characteristics. The trend, however, is limited only

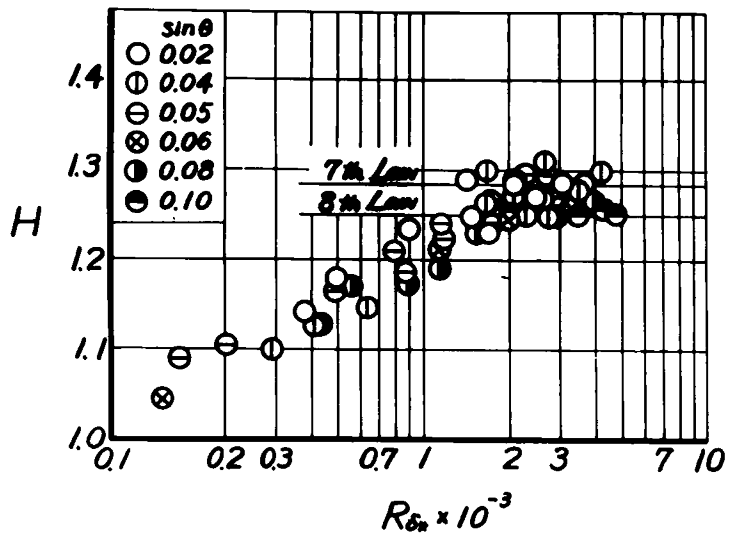

Fig. 1-11 Growth of H with increase of Reynolds number for displacement thickness

to the entrance reach where the guide vane will influence the shape of velocity profile of turbulent layer to some extent, and it is required the further study of this effect.

The power law is essentially empirical, so that the adequate consequence involved in the law may describe the actual phenomena in a very simple mathematical form and apply to the similar problem within the limit of applicability. The substantial characters, however, of mechanical principle of turbulence in flow itself is not concerned in this approach.

If the logarithmic form of velocity distribution is applied to the flow in turbulent layer, the constant of Kármán is expressed in a form of

$$\kappa = \sqrt{2C_f'} \cdot H/(H - 1), \tag{87}$$

and κ is regarded as 0.38 to 0.40 from the experimental results of steady uniform flows. Fig. 1-12 indicates the change of κ with the increase of Reynolds number for displacement thickness. Evidently in inspection of Fig. 1-12, the trend of κ describes a rapid decrease from a large value to a constant near 0.40 obtained by many investigators. The foregoing result of rapid decrease in κ is not explained and further study to this problem is also required.

(c) Mathematical Behaviours of Boundary Layer Equation in Open Channel Flows

The growth process of turbulent layer in open channel flows is largely influenced by the free surface and the channel characteristics, as shown in the foregoing, so that the required solution must be simultaneously solved by the three basic equations of (77), (82) and (83).

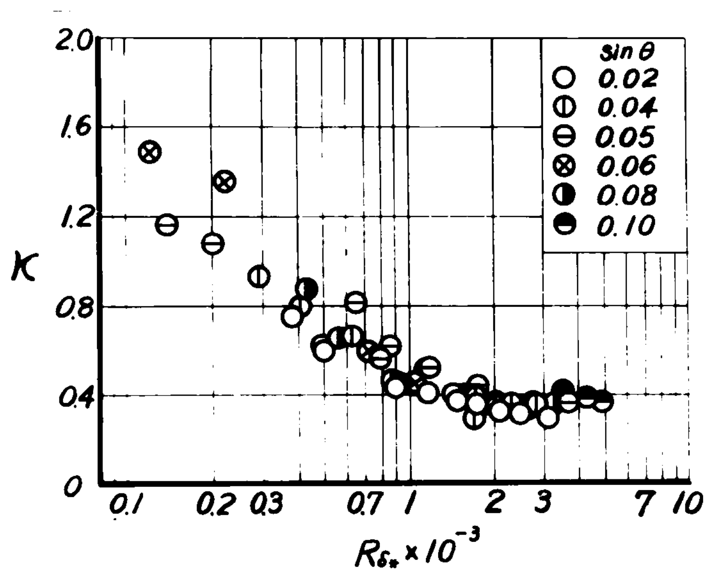

Fig. 1-12 Relationship between κ and Reynolds number for displacement thickness

Nevertheless, the usual procedure to estimate the boundary layer thickness in the open channel flow was based on the Prandtl-Schlichting equation of $(\delta/x) = 0.371/(u_0x/\nu)^{1/5}$ or similar equations modified by the actual flow condition in the potential flow[27].

The present author has studied the process of boundary layer growth by means of application of geometric theory in ordinary differential equation. Before discussing the mathematical behaviours of boundary layer growth, the hydraulic condition of layer at the initial point will be concerned. Selecting the origin of the coordinate system at $\delta = \delta_* = 0$, the flow condition is determined by

$$(u_0^2/2g) + (q\cos\theta/u_0) = \bar{H}_0, \qquad (88)$$

for given discharge and total head at the origin.

Eq.(88) has two possible roots of tranquil and shooting, and evidently two cases for the boundary layer growth will be observed in initially sub- and supercritical flows.

Among three equations of (77), (82) and (83), eliminating h and x, the relationship between the displacement thickness and the velocity in the main flow is obtained as follows.

$$\left\{1 - (HC_f/2\tan\theta)\right\}(d\delta_*/du_o) + \left\{(2 + H)/u_o - (1/H)(dH/du_o)\right\}\delta_*$$
$$= (HC_f/2\tan\theta)\left\{(u_o/g\cos\theta) - (q/u_o^2)\right\}. \qquad (89)$$

If the skin friction coefficient C_f and H determined by the velocity distribution in the layer are assumed constant throughout the whole zone under investigation, as an engineering approximation, Eq.(89) becomes linear and the solution is obtainable as a function of u_o.

The skin friction coefficient, however, can not be assumed constant and commonly a variable of Reynolds number as seen in the Blasius law, even if H is regarded to be a constant. If C_f is assumed proportional to the m th power of Reynolds number for displacement thickness,

$$C_f = 2\lambda(u_o\delta_*/\nu)^{-m}, \qquad (90)$$

and therefore, Eq.(89) is transformed into

$$(d\delta_*/du_o) = f_1(u_o, \delta_*)/f_2(u_o, \delta_*), \qquad (91)$$

in which

$$f_1(u_o, \delta_*) = (H\lambda/\tan\ )(u_o\delta_*/\nu)^{-m}\left\{(u_o/g\cos\theta) - (q/u_o^2)\right\} - (2 + H)\cdot(\delta_*/u_o),$$

and

$$f_2(u_o, \delta_*) = 1 - (H\lambda/\tan\theta)(u_o\delta_*/\nu)^{-m}.$$

Eq.(91) is evidently non-linear and must be commonly solved by the numerical analysis. The geometric characters of equation for the boundary layer growth is next treated, though the detailed treatment for analysis of hydraulic behaviours in the flow will be described in 1-2-4.

When $f_1(u_o, \delta_*)$ and $f_2(u_o, \delta_*)$ become simultaneously zero, Eq.(91) also becomes singular and the point is obtained as follows.

$$u_{oc}^3 = gq\cos\theta + (2 + H)g\nu\cos\theta(\tan\theta/H\lambda)^{-(1/m)}, \qquad (92)$$

and

$$\delta_{*c} = (\nu/u_{oc})(\tan\theta/H\lambda)^{-(1/m)}. \tag{93}$$

The location of singular point in the basic equation of turbulent layer is uniquely determined by given flow and channel characteristics. Fig. 1-13 indicates the location of singular point as a function of channel grade for the Blasius flow of a definite value of discharge. When the channel slope is steep, the singular point rapidly tends to critical for velocity and zero for displacement thickness. On the contrary, as the channel grade becomes mild to horizontal, the singular point increases its values in the supercritical regime and finally to infinity.

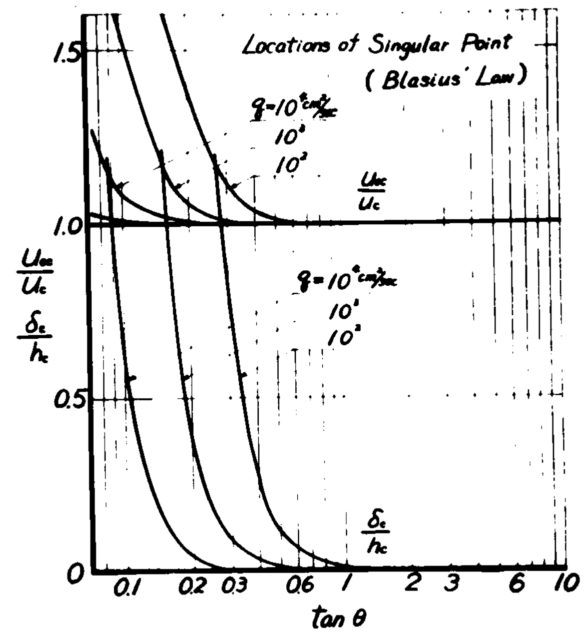

Fig. 1-13 Location of singular point of Blasius flows

The variation equation of Eq.(91) near the singular point is then

$$(d\delta_*/du_o) = (cu_o + d\delta_*)/(au_o + b\delta_*), \tag{94}$$

in which

$$a = m/u_{oc},$$
$$b = m/\delta_{*c},$$
$$c = (3/g\cos\theta) - (2 + H)(1 + m)(\delta_{*c}/u_{oc}^2),$$
$$d = -(2 + H)(1 + m)/u_{oc},$$

and u_{oc} and δ_{*c} indicate small values deviated from the singular point, as the origin of coordinate system has been transfered to the singular point.

The characteristic equation of Eq.(94) is

$$S^2 - [\{m - (2 + H)(1 + m)\}/u_{oc}]S - (3m/g\cos\theta) = 0, \qquad (95)$$

so that the discriminant

$$D = \{m - (2 + H)(1 + m)\}^2/u_{oc}{}^2 + (12m/g\cos\theta) > 0, \qquad (96)$$

is definitely positive and two real roots are of opposite sign, as $-(3m/g\cos\theta) < 0$. The singular point of Eq.(91) for the flow of power type in surface resistance is classified as a saddle point. The mathematical behaviour of turbulent layer growth in open channel flows is conclusively explained in the following.

The sign of a and b is evidently positive and that of d negative. The sign of c depends on

$$(u_{oc}^2/g\delta_{*c}\cos\theta) \gtreqless (2 + H)(1 + m)/3, \qquad (97)$$

and the upper and lower inequalities indicate $c \gtrless 0$. On the other hand, from Eqs.(92) and (93)

$$(u_{oc}^2/g\delta_{*c}\cos\theta) = 1 + H + (h_c/\delta_{*c}). \qquad (98)$$

The subtraction of Eq.(97) from Eq.(98) is

$$1 + H + (h_c/\delta_{*c}) - (2 + H)(1 + m)/3 = \{(H + 2)(1 - m) + (H - 1)\}/3 + (h_c/\delta_{*c}). \qquad (99)$$

As $H \geqq 1$, $1 > m > 0$, and $(\delta_{*c}/h_c) \rightarrow \infty$, so that the above relation indicates positive and thus c is definitely positive. The curve of $(d\delta_*/du_o) = 0$ is in the lower half plane divided by the curve of $(d\delta_*/du_o) = \infty$ for upstream reach from singular point and in the upper plane for downstream reach from it. The slope of $f_1(u_o, \delta_*) = 0$ at the singular point is positive and that of $f_2(u_o, \delta_*) = 0$ negative. A typical example of a family of curves of displacement thickness as a function of u_o is indicated in Fig. 1-14, for $q = 100$ cm^2/sec in a smooth channel of 1/500 in grade.

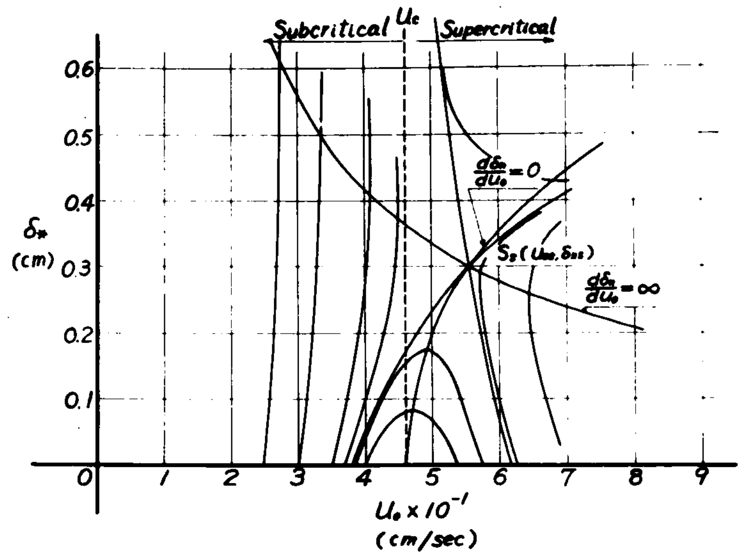

Fig. 1-14 Mathematical behaviours of turbulent boundary layer of Blasius flows

The hydraulic behaviour of turbulent layer growth in open channel flows is then as follows. At the origin δ_* is zero, and the abscissa indicates the initial value when the turbulent layer develops.

(1) When the initial velocity is tranquil, the layer develops rapidly to the free surface in the accelerative flows.

(2) As the initial velocity in subcritical regime approaches critical, so the layer still develops in the accelerative flow and reaches to the maximum value of δ_*. Thereafter the thickness decreases in the accelerative flow and finally it becomes again zero. This fact is very interesting in the boundary layer growth and first indicated by Craya and Delleur[8] in 1952.

(3) The initially supercritical flow produces the growth of boundary layer in the decelerative flow and especially when the initial value of supercritical flow is near critical, the layer attains its maximum value and then decreases.

(4) Two possible singular solutions, which pass the saddle point, as the boundary layer thickness indicate the continuous growth of layer, and one is in the initially subcritical flow and it develops in the accelerative flow and the other in the initially supercritical flow develops in the decelerative flow. These two

curves will describe the essential character of boundary layer growth in open channel flows and the possibility to play a primary role to discharge measurement in the future. It is still impossible to obtain the condition and make verifications by the experiments.

(5) When the channel grade becomes steep, the singular point approaches $(u_c, 0)$, as seen in Fig. 1-15. The slope of $f_1(u_o, \delta_*) = 0$ at the singular point approaches a constant of $3u_c/g\cos\theta \cdot (2 + H)(1 + m)$ and that of $f_2(u_o, \delta_*) = 0$ zero. Under such a condition, the geometrical representation of boundary layer growth is seen in Fig. 1-15. The initially subcritical flow only develops its layer in the decelerative flow, while for the initially supercritical flow the layer increases its thickness in the accelerative flow, which has been treated by the author[31], and explained in the next subsection.

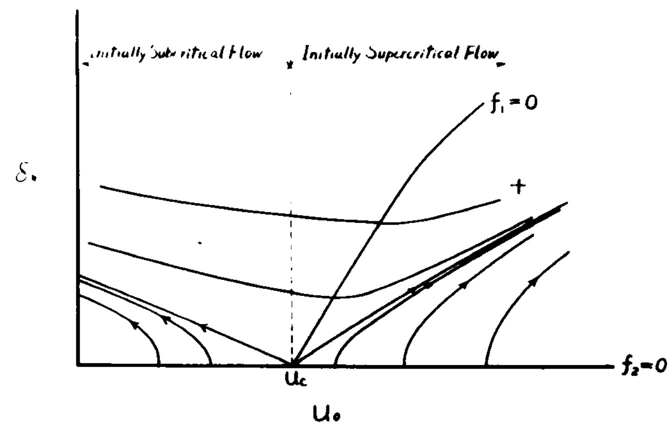

Fig. 1-15 Boundary layer growth in steep slopes

When the logarithmic law is used for the boundary layer resistance, the equation becomes

$$(du_o/dp) = u_o\{p^2 - 2p + 2 - (\kappa^2/\tan\theta)\}/\Big[(\kappa^2/C_s\tan\theta)(\exp -p)\{(u_o^3/g\cos\theta) - q\} - (\kappa^2/\tan\theta) - 2p^2 + 2p\Big], \tag{100}$$

in which p: $\kappa u_o/u^*$, C_s: $\nu/\kappa\exp(\kappa A)$ in Eq.(6) of 1-1-2.

In the same manner as in the foregoing case, the location of singular point is determined by

$$p_c = 1 \pm \sqrt{(\kappa^2/\tan\theta) - 1},$$

and

$$u_{oc}^3 = g\cos\theta\left[q + C_s e^{p_c}\{3 + (2\tan\theta/\kappa^2)(p_c - 2)\}\right], \qquad (101)$$

so that the channel grade is steeper than $\kappa^2 \doteq 0.16$, the equation has no singular point, while $\tan\theta < \kappa^2$, two singular points are in the u_o-p plane and classified under various flow conditions. It is however indicated that the Kármán constant is of basic significance to analyze the boundary layer growth and the further detailed study of the basic resistance law must be required for the advance of this problem.

(d) Critical Point of Boundary Layer Growth and Initiation of Free Surface Disturbance

The foregoing theory of boundary layer growth is applied until the layer spreads farther to the free surface. Denoting this point as the critical point as done by Halbronne, the location of the point has been studied with the relation to the initiation of free surface disturbance and the air-entrainment in steep chutes since Lane[26]. This theory can not imply the dynamics of free surface disturbance but may play a part of acceleration to the mechanism. In 1952, M.S. Priest and A. Baligh[35] investigated the initiation of free surface disturbance of open channel flows in steep chutes as seen in Fig. 1-16. The flow characteristics will be in the range of laminar flow, so that the laminar boundary layer growth will be approximately treated.

If the channel is of steep grade and the change of velocity and pressure head is assumed mainly due to the change of elevation, the following approximation is obtained from Eqs.(77) and (83).

$$dx\sin\theta = \{(u_o^2/g) - (q\cos\theta/u_o^2)\}du_o. \qquad (102)$$

With the use of laminar resistance law expressible by $\tau = (2\mu u_o/\delta)$, in which μ is the dynamic viscosity, the basic equation is linearized in a form of

$$(d\delta_*^2/du_o) + (9/u_o)\delta_*^2 = (10\nu/3g\sin\theta)\{1 - (gq\cos\theta/u_o^3)\}. \quad (103)$$

The solution of the above equation, which indicates the laminar layer growth, is

$$(3g\sin\theta/\nu)\delta_*^2 = u_o\{1 - (u_{oo}/u_o)^{10}\} - (10gq\cos\theta/7)(1/u_o^2)\{1 - (u_{oo}/u_o)^7\}, \quad (104)$$

where u_{oo} is the initial value of velocity at the origin.
For the accelerative flow, u_{oo}/u_o becomes rapidly zero, so approximately,

$$(3g\sin\theta/\nu)\delta_*^2 \doteqdot u_o - (10gq\cos\theta/7u_o^2). \quad (105)$$

At the critical point, $h_{cr} = \delta_{cr} = 3\delta_{*cr}$ and $u_{ocr} = 3u_m/2$, so that

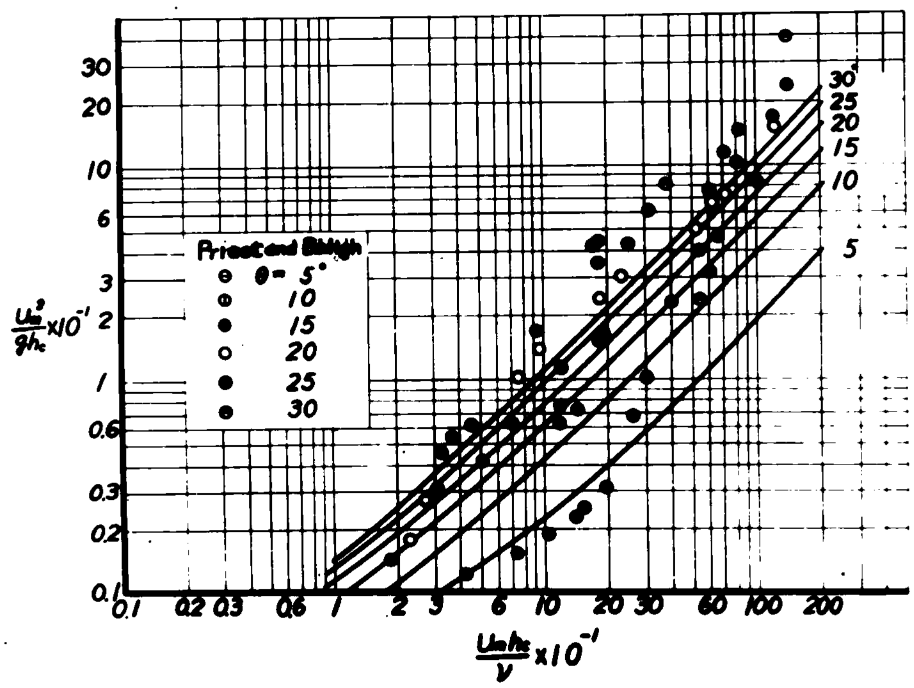

Fig. 1-16 Hydraulic characteristics of initiation of free surface disturbance in steep slopes

the critical condition will be expressible in terms of mean values of velocity and depth, as follows.

$$(27/8)u_m^3 = (21/28)(gu_m^2h_{cr}^2\sin\theta/\nu) + (40/28)gu_mh_{cr}\cos\theta ,$$

or

$$(u_m^2/gh_{cr}) = 0.423\cdot\cos\theta + 0.222\sin\theta(u_mh_{cr}/\nu). \qquad (106)$$

Eq.(106) is an expression of critical point of lamiar layer in terms of Reynolds and Froude numbers. Fig. 1-16 indicates the above relations as a parametric representation of channel grade. It is evidently seen that the foregoing approximation indicates a close agreement with the observation of Priest and Baligh and supposed the critical point will be a significant indication of free surface disturbance of open channel flows in steep chutes.

The same treatment will be considered to the turbulent layer of power type flow expressed by $C_f \propto (\nu/u_o\delta)^m$. The solution becomes approximately in the accelerative flow

$$(\tan\theta/\lambda H\nu^m)\delta_*^{(1+m)} = \left[(1 + m)/\beta + (1 + m)(1 + H)\right](u_o^{2-m}/g\cos\theta) - qu_o^{-(1+m)}/(1 + H), \qquad (107)$$

so that the critical condition for initiation of free surface disturbance is for the Blasius flow

$$(u_m^2/gh_{cr}) = 1.3733\cos\theta + 9.8552\cdot\sin\theta(u_mh_{cr}/\nu)^{(1/4)}. \qquad (108)$$

The condition is again expressible in terms of the Reynolds and Froude numbers for turbulent layer. The experimental verification, however, has not been made, and furthermore, the real channel, in which the problem is significant, is rough, so that the channel roughness will be of greater value and the relationship between the critical point and the actual channel will be discussed in the later part.

1 - 1 - 6 Hydraulic Behaviours of Moving Discontinuity in Open Channels

When a gate as a control structure in channels is suddenly opened or closed, rapid increase or decrease of discharge is produced, and the resulting discontinuous surface front will be travelled up and downstream. In general, any discontinuity through which the flow characteristics are changed in a moving fluid is called " discontinuous shock " or simply " shock ", although the term originally was used only for discontonuities in the compressible gas flow in aerodynamics. Hydraulic engineers usually call a moving discontinuity a bore and call a stationary one a hydraulic jump.

In past several sections, the flow is assumed continuous, whereas the present section deals with the basic behaviours of discontinuity, and therefore, the purpose is to derive the basic conditions of moving discontinuity in the physics of one dimensional hydraulics of open channel flows. Such attempts of deduction of the three conditions, mass and momentum conservations, and the rate of energy dissipation, have been seen in many hydraulic literatures for the case in which (1) the channel is horizontal and of rectangular shape, and (2) no resistive force is assumed to act, for the analysis of hydraulic transient problems in the classical theory of open channel flows.

In 1949, R.F. Dressler[36] refined to establish the discontinuous shock conditions in the case of two dimensional hydrostatic flows under the action of resistive force. More recently, in 1954, Iwagaki and the author[37] extended the conditions for all shapes of section in open channels to reveal the clear establishment of roll wave characteristics in steep chutes and their associated problems.

In hydraulic design of conveyance and control structures in open channels, often encountered is that a solution of transient behaviours of flow must be obtained for a particular channel geometry, so that the present section deals with the general conditions for a moving discontinuity shock in open channels. It will be understood in the later section that the deduction of three shock conditions is much convenient to discuss the general characters of transient problems of translation flows like surges and translation waves as an engineering approximation.

Let consider a discontinuity shock at abscissa $\xi(t)$ moving with the fluid particles. Since all water particles will be regarded to remain in a vertical section as the first approximation, it is possible to consider the fluid in a region bounded by the two vertical planes at $a_0(t)$ and $a_1(t)$, as the whole flow region moves to the running direction. The horizontal line -D is a reference line to which all potential energy under the gravitational action may be refered.

The laws of conservation of mass and momentum can be then applied to the enclose region of flow as the region moves with the course of time.

Mass conservation; $$\lim_{a_1 \to a_0} \cdot (d/dt) \iint_{a_1}^{a_0} \rho dA dx = 0, \tag{109}$$

Momentum conservation; $$\lim_{a_1 \to a_0} \cdot (d/dt) \iint_{a_1}^{a_0} \rho u dA dx = \int_{a_1} p dA - \int_{a_0} p dA - \int_{a_1}^{a_0} \tau s dx. \tag{110}$$

In the hydraulic treatment of fluid flows, the thermodynamic influences are not introduced to the flow, so that the rate of change of energy E is expressed as

$$(dE/dt) = \lim_{a_1 \to a_0} \cdot (d/dt) \iint \{(\rho u^2/2) + \rho g(y\cos\theta + z)\} dA dx + \int_{a_0} pu dA - \int_{a_1} pu dA + \int_{a_1}^{a_0} u_b \tau s \, dx. \tag{111}$$

As $a_1 \to a_0$, the last integral in Eqs.(110) and (111) will approach zero since the region approach zero while the integrand remains finite. It is, hence, of evidence that the body force will indicate no influence on the condition of a moving shock. In calculating the time derivatives of these integrals, the usual integral procedure may be used, as follows.

$$\lim_{a_1\to a_0}(d/dt)\iint_{a_1}^{a_0}\rho dAdx = \lim_{a_1\to a_0}\int_{a_1}^{a_0}(\partial/\partial t)(\rho A)dx + \lim\{|\rho A|_{\xi_-}(d\xi/dt) - |\rho A|_{a_1}(da_1/dt)\} + \lim\{|\rho A|_{a_0}(da_0/dt) - |\rho A|_{\xi_+}(d\xi_+/dt)\}$$
$$= \lim_{a_1\to a_0}\int_{a_1}^{a_0}(\partial/\partial t)(\rho A)dx + \rho A_1(V_W - u_1) - \rho A_0(V_W - u_0).$$

Finally, the mass conservation condition is

$$\rho A_0(V_W - u_0) = \rho A_1(V_W - u_1). \qquad (112)$$

in which, V_W is the absolute velocity as the vector sum of wave celerity and velocity of water particles.

In the same manner, the momentum conservation law is expressible in terms of flow and channel characteristics and it is

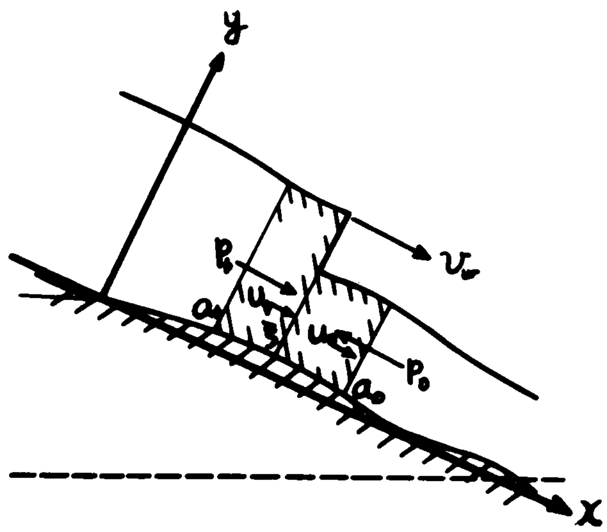

Fig. 1-17 Discontinuity shock moving in open channel flows

$$\rho u_0 A_0(V_W - u_0) - \rho u_1 A_1(V_W - u_1) = \int_{a_0} pdA - \int_{a_1} pdA. \qquad (113)$$

The expression for the rate of energy dissipation is also

$$(dE/dt) = \rho u_1^2 A_1(V_W - u_1)/2 - \rho u_0^2 A_0(V_W - u_0)/2 + \rho g\cos\theta A_1(h_1 - h_{G1})(V_W - u_1) - \rho g\cos\theta A_0(h_0 - h_{G0})(V_W - u_0) - \int_{a_1} pudA + \int_{a_0} pudA. \qquad (114)$$

If the multiplication of Eq.(113) by V_W on both sides and then the subtraction of Eq.(114) are made, the resulting equation becomes a general expression which is familiar to hydraulic engineers.

$$(dE/dt) = -A_o(V_w - u_o)\Big[\{(A_1 - A_o)/2A_oA_1\}\Big(\int_{a_1} pdA - \int_{a_o} pdA\Big) + \rho g\cos\theta \{(h_o - h_{Go}) - (h_1 - h_{G1})\}\Big]. \tag{115}$$

in which $h_{Gi} = h_i - y_{Gi}$.

In the analysis of foregoing cases, the pressure was assumed to be in an implicite form. The pressure of open channel flow, however, is usually assumed hydrostatic, if the curvature of stream lines is not appreciable, and these conditions for a discontinuity shock become

Mass conservation; $$M = -\rho A_o(V_w - u_o) = -\rho A_1(V_w - u_1). \tag{116}$$

Momentum conservation;

$$M\{(V_w - u_1) - (V_w - u_o)\} = \rho g\cos\theta(A_1 y_{G1} - A_o y_{Go}). \tag{117}$$

and the dissipation rate of energy is also calculated, and especially it becomes for two dimensional flows.

$$(dE/dt) = Mg\cos\theta(h_1 - h_o)^3/4h_oh_1, \tag{118}$$

and it is known as the Rayleigh condition.

The propagation of discontinuity shock will be concerned. Eliminating the velocity of disturbed flow, u_1, from Eqs.(112) and (113), the absolute velocity of shock front is

$$V_w = u_o \pm \sqrt{\{(A_1/\rho A_o)/(A_1 - A_o)\}\Big(\int_{a_1} pdA - \int_{a_o} pdA\Big)}\ . \tag{119}$$

When the wave height of disturbed wave is large compared with the depth of undisturbed flow, the shock front has the form of a breaking wave and the pressure is expressed by the hydrostatic law. Under such a condition, Eq.(119) becomes

$$V_w = u_o \pm \sqrt{g y_{Go}\cos\theta\{(A_1 y_{G1}/A_o y_{Go}) - 1\}/(1 - A_o/A_1)}\ . \tag{120}$$

This equation is very famous for hydraulic engineers and a basis for estimation of the velocity of surge front in open channels, though the influence of the boundary resistance, ignored in this theory, remains of minor importance.

Of great significance is that the absolute velocity for ascending waves becomes zero, and it is known as the formation of hydraulic jump. Consequently, the conjugate relationship between up- and downstream depths for hydraulic jump is

$$u_0^2/gy_{G0}\cos\theta = \{(A_1 y_{G1}/A_0 y_{G0}) - 1\}/(1 - A_0/A_1). \qquad (121)$$

The foregoing analysis for the propagation of surge front is mainly concerned for the hydrostatic flow in open channels. If the pressure distribution is extremely influenced by the vertical component of total acceleration and resulting curvature of free surface, the wave form is essentially expressed as a seris of surface undulation with cnoidal characteristics, as already briefly explained.

---

1) Boussinesq, J.V., Essai sur la théorie des eaux courantes, Mémoirs présentés par divers savants à l'Académie des Sciences, Paris, 1877.

2) Keulegan, G.H., and Patterson, G.W., Effect of Turbulence and Channel Slope on Translation Waves, Jour. Res. NBS, Vol. 30, June 1943.

3) de Saint Venant, Théorie du mouvement non permanent des eaux, avec application aux crues des riviers et à l'introduction des marées dans leur lit, C.R. 17, juillet 1871.

4) Fawer, C., Étude de quelques écoulements permanents à filets courbes, Thesis, Lausanne, 1937.

5) Sérre, F., Contribution à l'étude des écoulements permanents et variables dans les canaux, La Houille Blanche, Juin-Juillet, 1953.

6) Iwagaki, Y., On the Fundamental Equations for the Mean Flow of Water in Open Channels, Studies on the Thin Sheet Flow, 6 th Report, Proc. JSCE, Vol. 39, No. 10, Oct. 1954 (in Japanese).

7) Iwasa, Y., Analytical Considerations on Cnoidal and Solitary Waves, Memoirs, Fac. Eng., Kyoto University, Vol. 17, No. 3, Oct. 1955.

8) Craya, A., and Delleur, J.W., An Analysis of Boundary Layer Growth in Open Conduits near Critical Regime, Dept. Civ. Eng., Columbia University, CU-1-52-ONR-266, 1952.

9) Keulegan, G.H., Laws of Turbulent Flows in Open Channels, Jour. Res. NBS, Vol. 21, Dec. 1938.

10) Powell, R.W., Flows in a Channel of Definite Roughness, Trans. ASCE, Vol. 111, 1946.

11) Rouse, H., Elementary Mechanics of Fluid, John Wileys, New York, 1950.

12) Iwagaki, Y., Fundamental Study on Mechanism of Land Erosions by Rain-Water, Thesis, Kyoto University, Sept. 1955 (in Japanese).

13) Jaeger, C., Engineering Fluid Mechanics, Blackie, London, 1956.

14) Tsubaki, T., On the Formula of Velocity in Open Channels and Rivers, Report, Fluid Mech. Inst., Kyushu University, Vol. 4, No. 2, 1947 (in Japanese).

15) Hayami, S., On the Mechanics of Flow in a Wide Alluvial River, Shanghai Sci. Inst., July 1939.

16) Gebelein, H., Die Turbulenz, Springer, Berlin, 1935.

17) Rotta, J., Das in Wandnähe gültige Geschwindigkeitsgesetz turbulenter Strömungen, Ingenieur-Archiv, Band 18, 1950.

18) Clauser, F.H., The Turbulent Boundary Layer, Advances in Applied Mechanics, Vol. 4, Academic Press, New York, 1956.

19) Iwagaki, Y., On the Laws of Resistance to Turbulent Flow in Open smooth Channels, Proc. 2 nd Japan Nat. Cong. Appl. Mech., 1952.

20) Jegorow, S.A., Turbulente Überwellenströmung (Schiessen) im öffenen Gerinne mit glatten Wanden, Wasserkraft und Wasserwirtschaft, Heft 3, 1940.

21) Homma, M., Hydraulics (Fluid Mechanics for Hydraulic Engineers), Maruzen, Tokyo, 1952 (in Japanese).

22) Favre, H., Contribution à l'étude des courants liquides, Dunod, Paris, 1933.

23) DeMarchi, G., Canali con portata progressivamente cresiente, Energia Eletrica, July 1941.

24) Friedlich, K.O., On the Derivation of the Shallow Water Theory, Appendix to The Formation of Breakers and Bores by J.J. Stoker, Comm. Pure and Appl. Math., Vol. 1, 1948.

25) Keller, J.B., The Solitary Wave and Periodic Waves in Shallow Water, Comm. Pure and Appl. Math., Vol. 1, 1948.

26) Lane, E.A., Recent Studies on Flow Conditions in Steep Channels, Eng. News Rec., Jan. 2 1936.

27) for example, IAHR and ASCE, Proc. Minnesota International Hydraulics Convention, Sept. 1953.

28) Halbronn, G., Étude de la mise en régime des écoulements sur les ouvrages à forte pente, La Houille Blanche, Jan.-Feb. 1952.

29) Bauer, W.J., Turbulent Boundary Layer on Steep Slopes, Trans. ASCE, 1954.

30) Delleur, J.W., The Boundary Layer Development on a Broad Crested Weir, Proc. 4 th Midwestern Conference on Flui. Mech., Purdue University, Lafayette, Indiana, 1955.

31) Iwasa, Y., Boundary Layer Growth of Open Channel Flows on a Smooth Bed and Its Contribution to Practical Application to Channel Design, Memoirs, Fac. Eng., Kyoto University, Vol. 19, No. 3, July 1957.

32) Iwasa, Y., Boundary Layer Growth and Its Related Problems in Open Channel Hydraulics, Abstract, 3 rd Conference for Hyd. Res., JSCE, May 1958 (in Japanese).

33) Iwasa, Y., Contributions of Boundary Layer Theory to Flows in Open Channels, Proc. 1 st Symposium for Hyd. Res., Japan Academic Conference, Apr. 1959.

34) Ishihara, T., Iwagaki, Y., and Goda, T., Studies on the Thin Sheet Flow, Trans. JSCE, No. 6, Aug. 1951 (in Japanese).

35) Priest, M.S., and Baligh, A., Free Surface Instability of Liquid in Steep Channels, Trans. AGU, Feb. 1954.

36) Dressler, R.F., Mathematical Solution of the Problem of Roll-Waves in Inclined Open Channels, Comm. Pure and Appl. Math., Vol. 2, No. 2/3, 1949.

37) Iwagaki, Y., and Iwasa, Y., On the Hydraulic Characteristics of the Roll-Wave Trains, Studies of the Thin Sheet Flow, 7 th Report, Jour. JSCE, Vol. 40, No. 1, Jan. 1955 (in Japanese).

2. General Theory of Transitional Characteristics of Steady Flows in Channel Transitions and Controls

1 - 2 - 1 Basic Features of Steady Flows

In the design practice of common hydraulic structures like conveyance structures, in which the flow is usually classified as a gradually varied flow, and control structures, which induce the flow rapidly varied, the basic principle is that the discharge is independent of time, though special problems in floodway projects aimed to carry safely the design flood discharge are subjected to variable rates of discharge with time. The hydraulic characteristics of steady flows, therefore, are independent of time and vary from point to point.

The mathematical and hydraulic studies of steady flows, initiated as far as back 18 th century, have been progressed, with the contribution to the real success in hydraulic engineering, by a large number of engineers and scientists. The time-independent behaviours of flows, which are one of the most important branch in the hydraulics of open channel flows, are practically divided into two parts of gradually varied flow influenced by the major importance of surface resistance and of rapidly varied flow* characterized by the appreciable change in flow characteristics produced by the local change in channel geometry. Both types of flows, however, are resulted from the basic principles of motion described in the previous chapter, and the final purpose of the study of steady flows is to reveal the surface profiles and resulting character-

---

Definition of rapidly varied flow described in this study is a little different from that of Jaeger. His definition is the flow which can not be analyzed by means of the energy theorem alone, whereas in this study, the flow which changes rapidly its characteristics may be called as rapidly varied.

istics of flow in the one dimensional method of hydraulic procedure. In the gradually varied flow of classical hydraulics, the change of flow characteristics and channel geometry is so gradual that the surface profile will be estimated with sufficient accuracy in engineering practice, and many investigators[1] have been enforced to reveal the hydraulic characteristics of flow. On the other hand, the rapidly varied flow as a counterpart of gradually varied flow is resulted by the abrupt change in flow and channel characteristics. The method of energy analysis in classical hydraulics is insufficient to make the formulation of surface profiles, and the application of momentum theorem to establish the empirical relationship between flow characteristics and channel geometry, with the aid of the critical depth theory as a function of discharge, energy and momentum flux for particular channels, prevails.

The physical behaviours of steady flows in conveyance and control structures are determined as solutions obtained by all methods of analysis for flows, so that a universal procedure to analysis must be established for the further progress in the study of steady flows. The problem involved thus entails the basic principles of motion of steady flows in the one dimensional procedure described in the previous chapter and the mathematical properties of the equation expressing the physics of steady flows, which will be definitely explained in the later part of this paper as transitional characteristics of steady flows in open channels.

Before concerning such transitional behaviours of surface profiles of water in the light of modern knowledge of applied mathematics, this chapter commonly treats with the basic characteristics of time-independent flows. The most important problem in the hydraulics of steady flows is the concept of critical and normal depths in the basic equation, so that the first section concerns with the hydraulic significance and behaviours of such depths. Of special interesting problems in engineering purpose is the case in which the normal depth becomes equal to the critical depth for a particular non-uniform channel geometry and a boundary characteristic. This problem is the key stone for the modern trend in hydraulic research of steady flows, though the usual procedure is based on the condition that a singular point exists at infinity. The basic treatment, therefore, will be followed and in the final section of this chapter the mathematical proof of Bélanger theorem, Böss theorem and the generalized Jaeger's theory on the simultaneity of maximum discharge and minimum energy will be explained.

As has been explained in the foregoing, the steady state characteristics of open channel flows are widely studied by a large number of scientists and engineers, and especially the hydraulics of gradually varied flows in uniform channel is completely advanced. In reality, however, nearly almost all channels involve channel transitions and controls with non uniformity in cross section, channel grade and roughness, so that the engineering contribution of fruitful investigations developed by many hydraulic engineers since the 18 th century to actual problems in hydraulic design is extremely restricted. The general theory of transitional characteristics of steady flows which will be described in this chapter is essentially of mathematical form and gives hydraulic engineers the basic information of steady flow hydraulics. More precisely, in the hydraulics of gradually varied flows, the general rule in determination of surface profiles of water will be indicated, and in the hydraulics of rapidly varied flows, the hydraulic performance of control structures in hydraulic works will be illustrated. The transitional characteristics of flow, therefore, are the most important essentials in the hydraulics of steady flows, and it is surely hoped for the author that the future development of hydraulic research projects in a realm of steady flow hydraulics will be born from the theory described herein.

1 - 2 - 2 Normal and Critical Depths of Steady Flows

If the channel is completely uniform, the physical regime of steady flow will finally become uniform. The resistive and gravity forces are also in exact balance and the resulting surface profile is parallel to the channel bed. The condition of flow is called normal or uniform and the hydraulic characteristics are determined by the common resistance law like Chézy and Manning formulas. The normal depth is evidently a function of the shape, the roughness, its slope and the given rate of discharge.

If the channel geometry is non-uniform, the uniform flow,in which the free surface is parallel to the channel bottom, is influenced by the non-uniformity of channels. The surface gradient of steady flows by means of the energy approach is obtained by Eqs.(68) and (69) in 1-1-4 and it is

$$(dh/dx) = f_1(x, h)/f_2(x, h), \tag{1}$$

in which

$$\begin{aligned} f_1(x, h) &= \sin\theta - (\tau/\rho gR)(u_b/u_m) - (\partial H_0/\partial x), \\ &= \sin\theta - (\tau/\rho gR)(u_b/u_m) + (\alpha Q^2/gA^3)(\partial A/\partial x) - (Q^2/2gA^2)(\partial\alpha/\partial x) \\ &\quad - h\cos\theta(\partial\lambda/\partial x) + \lambda h\sin\theta(\partial\theta/\partial x), \end{aligned} \tag{2}$$

and

$$\begin{aligned} f_2(x, h) &= (\partial H_0/\partial x), \\ &= \lambda\cos\theta + h\cos\theta(\partial\lambda/\partial h) - (\alpha Q^2/gA^3)(\partial A/\partial h) + (Q^2/2gA^2)(\\ &\quad \partial\alpha/\partial h), \end{aligned} \tag{3}$$

so that the uniform condition in flow as a solution of $f_1(x, h) = 0$

if $f_2(x, h) \neq 0$ is largely different from that in uniform channels. Though P. Massé[2] called $f_1(x, h) = 0$ as the curve of quasi-normal flows and F.F. Escoffier[3] the transition curve, the depth obtained by $f_1(x, h) = 0$ is called normal in this study and it is evidently a variable of distance as well as channel and flow characteristics.

In the same manner, the uniform depth is also defined by the momentum approach in the one dimensional analysis.

In the hydraulic practice as an engineering approximation, α and λ are assumed nearly constant, and thus all derivatives of α and λ with respect to distance and depth become zero. Furthermore, in gradually varied flows, the pressure is also assumed hydrostatic, and Eq.(1) is reduced to the well known equation of gradually varied flows in non-uniform channels as seen in many hydraulic literatures.

If the denominator in Eq.(1) becomes zero, the surface gradient also becomes quite large to infinity. The critical depth of steady flow in open channels is defined as a solution of Eq.(3) in the energy approach and the similar conclusion will be obtained in the momentum approach. In the basic equation for surface gradient, a point at which the denominator is zero while the numerator is not zero is a singular point of steady flow equation, and therefore the mathematical significance of critical depth for the behaviour of surface profiles will be evident.

The critical regime of steady flows is also determined by $(\partial H_0/\partial h) = 0$ and $(\partial M_0/\partial h) = 0$, which indicate the minimum energy and momentum theorems for particular discharges. The minimum energy theorem is known as the Böss theorem[4], which was presented in 1919. In the past these minimum theorems in energy and momentum flux were commonly used for the analysis of steady flows and especially of rapidly varied flows by hydraulic engineers.

In the channel with particular geometry, the specific energy and the momentum flux are defined by Eqs.(68) and (75) in the previous chapter, so that the critical condition is

$$A_c^3\{1 + (h_c/\lambda_c)(\partial\lambda/\partial h)_c\} = (Q^2/\lambda_c g\cos\theta_c)\{\alpha_c(\partial A/\partial h)_c - (A_c/2)\cdot(\partial\alpha/\partial h)_c\}, \tag{4}$$

for the energy approach, and

$$A_c\{(\partial/\partial h)(A\dot{y}_G)\}_c = (Q^2/\lambda_c' g\cos\theta_c)\{(\beta_c/A_c)(\partial A/\partial h)_c - (\partial\beta/\partial h)_c - (y_{Gc}A_c^2/\lambda_c')(\partial\lambda'/\partial h)_c, \tag{5}$$

for the momentum approach. If the idealized condition that α, β, λ and λ' are constant and the channel geometry is rectangular is assumed in the above equations, it is evidently understood that the resulting critical depth is of familiar type for hydraulic engineers.

In view of the critical depth theorem, the most significant relationships are known as the Bélanger theorem indicated the maximum discharge principle for a particular energy head and the generalized theory of Jaeger for the simultaneous occurrence of maximum discharge and minimum energy. The proof of these relationships and their hydraulic significance will be explained in the later section of this chapter.

1 - 2 - 3 Hydraulic Characteristics of Tranquil and Shooting Flows

(a) Basic Features of Two Regimes of Tranquil and Shooting in Steady Flows

Before starting the discussion on the basic characters of transitional behaviours of flow by means of the application of geometric theory of ordinary differential equation, two physical regimes of tranquil and shooting in the open channel flow will be briefly explained.

If the flow depth at a point is greater than the critical depth described in Eqs.(4) or (5), the flow is classified as subcritical in this study, and on the contrary, the supercritical regime of flow indicates the flow depth is less than the critical depth. Apparently, in the minimum energy and momentum theorems in the previous section, two real values of subcritical and supercritical for a given discharge are obtained for particular specific energy and momentum flux. This relationship will be readily understood in Fig. 1-18, which is constructed for the most simplified flow pattern of hydrostatic flow in uniform channels. The critical condition dividing two regimes of flow is indicated at h_c, H_{oc} and M_{oc}.

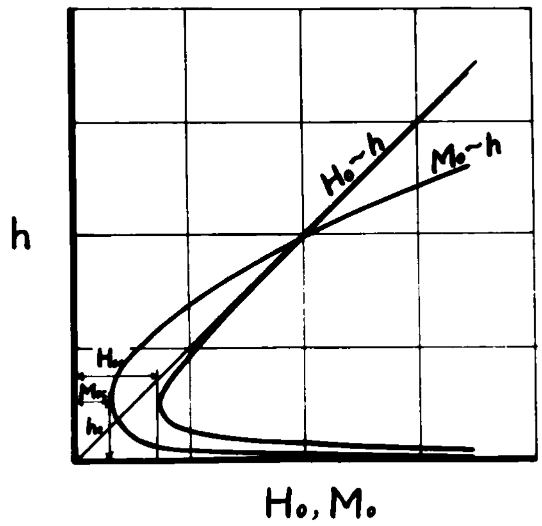

Fig. 1-18 Relationship between depth, energy and momentum flux

If Eqs.(4) and (5) are plotted in the same figure, this condition will be not coincided, owing to the approximate expression in the basic principles of open channel flows. The critical condition obtained by both approaches of energy and momentum must be coincided, if the physics of flow is exactly represented by the mathematical form.

Two values of subcritical and supercritical for particular values of specific energy are known as the alternate depths, corresponding to two sections of a channel transition for which the loss of head by the surface resistance is negligible as an engineering approximation, and those for a particular momentum flux are called as the conjugate or sequent depths corresponding to the up- and downstream depths of the hydraulic jump.

Of most significance in the two regimes of open channel flows is that very small disturbances in flow characteristics like velocity and depth in tranquil flow may be transmitted upstream, whereas such disturbances in shooting flows can not influence the flow upstream from the boundary change of critical condition. This important theorem for the open channel flow can not be proved by the rigorous analysis, because of the great difficulty to derive the universal open channel equation in a curved flow. The author has treated with rather general explanation for the non-hydrostatic flow over a mild curved boundary which will be indicated in the following.

The approximate basic equation of the unsteady flow by means of the energy approach is obtained by Eq.(63) in the previous chapter.

$$(\beta/g)(\partial u_m/\partial t) + (\alpha u_m/g)(\partial u_m/\partial x) + \lambda\cos\theta(\partial h/\partial x) + \{(\beta - \alpha)/2g\,(u_m/A) + h\cos\theta(1 - \lambda)/u_m A\}(\partial A/\partial t) + (u_m^2/2g)(\partial\alpha/\partial h)(\partial h/\partial x) + (u_m/2g)(\partial\beta/\partial h)(\partial h/\partial t) + h\cos\theta(\partial\lambda/\partial h)(\partial h/\partial x) = \sin\theta - (\tau/\rho g R)(u_b/u_m), \tag{6}$$

in which α, β and λ are assumed variables of depth.

Let consider the small disturbance travelling up- and downstream on the free surface of uniform flow. With the use of the equation of continuity, the linearized version of Eq.(6) becomes

$$R(\partial^2 h'/\partial x^2) + S(\partial^2 h'/\partial x\partial t) + T(\partial^2 h'/\partial t^2) + \cdots\cdots = 0, \tag{7}$$

where

$$R = (\lambda_0 g\cos\theta/gA_0^3)\left[A_0^3\{1 + (h_0/\lambda_0)(\partial\lambda/\partial h)_0\} - (Q^2/\lambda_0 g\cos\theta)\{\alpha_0(\partial A/\partial h)_0 - (A_0/2)(\partial\alpha/\partial h)_0\}\right],$$

$$S = h_0\cos\theta(1 - \lambda_0)(1/Q)(\partial A/\partial h)_0 - (Q/2gA_0^2)(\partial A/\partial h)_0(\beta_0 + 3\alpha_0) + (Q/2gA_0)(\partial\beta/\partial h)_0,$$

and

$$T = -(\beta_0/gA_0)(\partial A/\partial h)_0.$$

If Eq.(7) is of hyperbolic type, the absolute velocity of a small disturbance is derived by the characteristic equation of Eq.(7), and it is

$$V_w = (S \pm \sqrt{S^2 - 4RT})/2T. \tag{8}$$

As the absolute velocity of disturbance is the sum of the celerity and the undisturbed fluid velocity so that a channel section where the flow velocity is equal to the ascending wave celerity acts as a barrier to the transmittal. The critical condition yields R = 0, and the flow area of the section is

$$A_0^3\{1 + (h_0/\lambda_0)(\partial\lambda/\partial h)_0\} = (Q^2/\lambda_0 g\cos\theta)\{\alpha_0(\partial A/\partial h)_0 - (A_0/2)(\partial\alpha/\partial h)_0\}.$$

This is equivalent to the critical condition for steady flows, and the curve of critical depth for particular channels, therefore, indicates the locus of such barriers. This physical condition of sub- and supercritical flows is also of significance in the computation of water surface profiles and the routing of tranquil branch proceeds upstream while the analysis of shooting branch proceeds downstream. The verification of this theorem for open channel flows by means of the momentum approach will be seen in the next subsection.

The main characteristics of tranquil flows with low kinetic energy are described as the small changes of velocity head are of great importance owing to that the flow is sustained by low gradient. When the flow becomes critical, the stage variation with the change of head and momentum flux are pronounced. Furthermore, the influence of vertical acceleration becomes appreciable, and therefore, the basic equation in the gradually varied flow also is influenced by the non-hydrostatic pressure due to the surface curvature, and the resulting formation of surface undulation is observed as Boussinesq, Fawer, Sérre and the author did.

The shooting flow is generally characterized by the fact that large variations in head and momentum flux are reflected little in the value of depth, since they are due almost entirely to change in the kinetic portion of the head, and in the shooting flow many outstanding features not found for tranquil flows are pronounced. Small changes in boundary alignment produce shock waves and the solid boundary makes the flow unstable to the formation of roll waves, known as a result of hydraulic stability[5)].

(b) Hydraulic Stability in Supercritical Flows

In long chutes, the progressive roll waves in the shooting flow as a train are often observed. The formation of roll waves is essentially resulted from the hydraulic instability and this fact has been verified by H. Jeffreys[6)], G.H. Keulegan and G.W. Patterson[7)], and Iwagaki and the author[8)], though V. Cornish[9)] first observed. It should be noticed that this instability is different in its essential character from the hydrodynamic stability that leads to the transition from laminar to turbulent flows and it presupposes the fluid indicates the transition from steady uniform flows with a plane smooth surface to different flows with ripples and transverse ridges.

In 1925, Jeffreys studied this criterion for initial instability of flows with the Chézy type of resistance formula and obtained the critical condition was 2 in Froude number. Keulegan and Patterson derived the stability criterion for the Manning flow with the aid of the concept of wave celerity of volume element of Boussinesq and the Froude number for stability was 1.5. Recently, V.V. Vedernikov[10)] obtained the generalized criterion, which is known as the Vedernikov number by another approach with rather complicated form in mathematical treatment. A. Craya[11] analyzed the time growth of an infinitesimally small shock wave and obtained the criterion for the coefficient of resistive force $\lambda = KR^{\rho}$ in 1952. R.F. Dressler and F.V. Pohle[12] have refined the common procedure to evaluate the time growth of small disturbances in two dimensional flow and the author[5] also extended the stability criterion to the flow in all shapes of channels and verified the Vedernikov number.

It is of common observation that the unstable disturbance produced becomes a train of roll waves with definite periodic wave patterns, so that the instability criterion is considered exactly equivalent to the formulation condition of roll waves by many investigators. On the other hand, H. Thomas[13], Dressler[14], and T. Ishihara, Iwagaki and the author[8], [15] investigated the hydraulic characteristics of roll waves that might be reduced to be stationary when observed from the moving coordinate system with the constant absolute velocity, and verified these two conditions were equivalent to each other.

If the same procedure of analysis as in the method of hydrodynamic stability are applied to the hydraulic stability problems, the procedure will be divided into two methods of approach. One is apparently the energy method in which the time growth or decay of energy of disturbed wave will be treated, and the other is to investigate the initiation condition of continuous time growth of a very small disturbance imaginary superposed on the fluid surface, as Jeffreys first did for the Chézy flow. In this subsection, the author's present purposd is to establish a general form of the criterion for initial instability of shooting flow, and first will be treated the condition for the general form known as the Vedernikov number and the influence of the character of disturbed wave is followed.

If the channel bottom is assumed not to be abruptly varied, the basic relationships of momentum, which involves the simple expression for the surface resistance compared with that in the energy approach, and continuity are

$$(1/g)(\partial u_m/\partial t) + (\beta u_m/g)(\partial u_m/\partial x) + (1-\beta)(u_m/gA)(\partial A/\partial t) + \{\lambda''\cos\theta + (u_m^2/g)(\partial\beta/\partial h)\}(\partial h/\partial x) = \sin\theta - (\tau/\rho g R), \tag{9}$$

and

$$(\partial A/\partial t) + u_m(\partial A/\partial x) + A(\partial u_m/\partial x) = 0, \tag{10}$$

in which

$$\lambda'' = 1 + \int(\partial/\partial x)(\Delta p/\rho g)dA/A\cos\theta(\partial h/\partial x).$$

Let consider small disturbances of velocity and depth expressible by u' and h', and thus local values of velocity and depth in the disturbed flow are

$$u_m = u_{mo} + u', \text{ and } h = h_o + h', \tag{11}$$

in which the subscript o indicates the value under steady uniform condition. Substituting Eq.(11) into Eqs.(9) and (10), and differentiating Eq.(9) with respect to the distance and Eq.(10) to the time, the linearized version of Eqs.(9) and (10) becomes as follows.

$$(1/g)(\partial^2 u'/\partial x\partial t) + (\beta_o u_{mo}/g)(\partial^2 u'/\partial x^2) + (1-\beta_o)(u_{mo}/gA_o)(\partial A/\partial h)_o \cdot(\partial^2 h'/\partial x\partial t) + \{\lambda_o''\cos\theta + (u_{mo}^2/g)(\partial\beta/\partial h)_o\}(\partial^2 h'/\partial x^2) = \{(\tau_o/\rho g R_o^2)(\partial R/\partial h)_o - (\partial\tau/\partial h)_o(1/\rho g R_o)\}(\partial h'/\partial x) - (1/\rho g R_o)(\partial\tau/\partial u)_o(\partial u'/\partial x),$$

and

$$A_o(\partial^2 u'/\partial x\partial t) + u_{mo}(\partial A/\partial h)_o(\partial^2 h'/\partial x\partial t) + (\partial A/\partial h)_o(\partial^2 h'/\partial t^2) = 0,$$

where all higher terms depending on squares and products of small disturbances are ignored and the following second order linear partial differential equation will be derived.

$$\{\lambda''_0\cos\theta - (\beta_0 u_{mo}^2/gA_0)(\partial A/\partial h)_0 + (u_{mo}^2/g)(\partial\beta/\partial h)_0\}(\partial^2 h'/\partial x^2) - (2\beta_0 u_{mo}/gA_0)(\partial A/\partial h)_0(\partial^2 h'/\partial x\partial t) - (1/gA_0)(\partial A/\partial h)_0(\partial^2 h'/\partial t^2) + (1/\rho gR_0)(\partial\tau/\partial h)_0 - (\tau_0/\rho gR_0^2)(\partial R/\partial h)_0 - (u_{mo}/\rho gA_0R_0)(\partial\tau/\partial u)_0(\partial A/\partial h)_0(\partial h'/\partial x) - (1/\rho gR_0A_0)(\partial\tau/\partial u)_0(\partial A/\partial h)_0(\partial h'/\partial t) = 0. \tag{12}$$

This is the basic equation in the investigation of hydraulic condition for initial instability in the present purpose.

As the following inequality of

$$(\beta_0^2 - \beta_0)(u_{mo}^2/g^2A_0^2)(\partial A/\partial h)_0^2 + (1/gA_0)(\partial A/\partial h)_0\{\lambda''_0\cos + (u_{mo}^2/g)(\partial\beta/\partial h)_0\} > 0, \tag{13}$$

will be obtained, so the basic equation is of hyperbolic type, and the absolute velocity V_w of small disturbed wave is obtained by means of its characteristics.

$$V_w = \left[\beta_0 \pm\sqrt{\beta_0^2 - \beta_0 + \{s_0/F_r^2(dA/dh)_0\}\{\lambda''_0 + F_r^2R_0(\partial\beta/\partial h)_0}\right]u_{mo}. \tag{14}$$

The upper positive sign indicates the velocity of a descending wave and the lower negative sign that of an ascending wave. Although the usual method to evaluate the absolute velocity of a small wave is derived by the small amplitude theory, the above characteristic approach is rather simple. The ratio of velocity of undisturbed flow to the celerity thus becomes

$$\frac{(V_w - u_{mo})/u_{mo}}{} = \beta_0 - 1 \pm\sqrt{\beta_0^2 - \beta_0 + \{s_0/F_r^2(dA/dh)_0\}\{\lambda''_0 + F_r^2R_0(\partial\beta/\partial h)_0\}}. \tag{15}$$

This equation also is an important relationship of very small waves and often used in the later section. Special cases in Eqs.(14) and (15) are reduced to the well known relationship in the classical theory of hydraulics.

If the absolute velocity of an ascending wave becomes zero, the disturbance can not travel to the upstream direction and the critical condition is

$$\lambda_o''\cos\theta - (\rho_o u_{mo}^2/gA_o)(\partial A/\partial h)_o + (u_{mo}^2/g)(\partial\rho/\partial h)_o = 0. \quad (16)$$

If the relation of $\int(\partial/\partial x)(p/\rho g)dA = (\partial/\partial x)\int(p/\rho g)dA$, which is valid for uniform channels, is assumed,

$$\lambda_o'' = 1 + \{(\partial/\partial x)\int(p/\rho g)dA - A_o\cos\theta(\partial h/\partial x)\}/A_o\cos(\partial h/\partial x)$$
$$=(1/A_o\cos\theta)(\partial/\partial h)(\lambda_o' A_o y_{Go}\cos\theta) = (1/A_o)(\partial/\partial h)(\lambda_o' A_o y_{Go})$$
$$=(\lambda_o'/A_o)\{(\partial/\partial h)(Ay_G)\}_o + y_{Go}(\partial\lambda'/\partial h)_o, \quad (1\ )$$

and inserting Eq.(17) into Eq.(16), the final expression is

$$A_o\{(\partial/\partial h)(Ay_G)\}_o = (Q^2/\lambda_o' y_{Go}\cos\theta)\cdot\{(\rho_o/A_o)(\partial A/\partial h)_o - (\partial\rho/\partial h)_o$$
$$- (y_{Go}A_o^2/\lambda_o')(\partial\lambda'/\partial h)_o.$$

This equation also indicates the critical condition of steady flows by means of the momentum approach and verifies the critical depth is a barrier to transmittal of disturbance.

The initiation condition of hydraulic instability will be treated as follows. Let consider the time growth or decay of an infinitesimal small disturbed wave expressed by

$$h' = A\exp(\gamma t + iL'x), \quad (18)$$

where $\gamma = r + iT'$ and the real part of γ indicates the variation of amplitude of a disturbed wave with respect to the time. The instability condition therefore depends on the sign of the real part of γ. That is

$$r = \mathcal{R}(\gamma) = 0 \quad \begin{array}{l}\text{.......... unstable}\\ \text{.......... stable}\end{array} \quad (19)$$

and $\mathcal{R}(\gamma) = 0$ represents also the limiting condition between two states of flow. The insertion of Eq.(18) into the basic equation of (12) then yields the following relation between γ and L'.

$$(1/A_o)(dA/dh)_o\gamma^2 + 2\{(1/2\rho A_o R_o)(dA/dh)_o(\partial\tau/\partial u)_o + iL'(\rho_o u_{mo}/$$

$A_0)(dA/dh)_0\}\gamma + L'^2\{\lambda_0''g\cos\theta - (\beta_0 u_{mo}^2/A_0)(dA/dh)_0 + u_{mo}^2(\partial\beta/\partial h)_0 + iL'\{(u_{mo}/\rho A_0 R_0)(dA/dh)_0(\partial\tau/\partial u)_0 - (1/\rho R_0)(\partial\tau/\partial h)_0 + (\tau_0/\rho R_0^2)(dR/dh)_0\} = 0.$

Solving for γ becomes

$$\gamma = -\{(1/2\rho R_0)(\partial\tau/\partial u)_0 + iL'\beta_0 u_{mo}\} \pm \sqrt{\{(1/2\rho R_0)(\partial\tau/\partial u)_0 + iL'\beta_0 u_{mo}\}^2 + \{A_0/(dA/dh)_0\}[L'^2\{\lambda_0''g\cos\theta - (\beta_0 u_{mo}^2/A_0)(dA/dh)_0 + u_{mo}^2(\partial\beta/\partial h)_0\} + iL'\,(u_{mo}/\rho A_0 R_0)(dA/dh)_0(\partial\tau/\partial u)_0 - (1/\rho R_0)(\partial\tau/\partial h)_0 + (\tau_0/\rho R_0^2)(dR/dh)_0\}]}. \tag{20}$$

With the use of Eqs.(15) and (19), the following relation is finally obtained.

$$- A_0\{(\partial\tau/\partial A)_0 - (\tau_0/A_0)(dR/dA)_0\}/(\partial\tau/\partial u)_0 \gtreqless V_w - u_0. \tag{21}$$

Eq.(21) is the expression of initiation condition for hydraulic instability in open channel flows. As seen in this equation, however, the hydraulic description of condition seems rather unfamiliar to engineers, so the better expression in terms of channel characteristics like channel geometry and resistance law is indicated.

In laminar flows, the velocity profile is assumed parabolic and thus β_0 becomes 1.2 as Ishihara, Iwagaki and Goda[16] verified through their research, so that the instability criterion for the laminar flow becomes

$$2Mu_{mo}/(V_w - u_{mo}) \gtreqless 1, \tag{22}$$

in which M is the shape parameter. Or expressing in terms of the Froude number,

$$F_r^2 \gtreqless \{s_0/(dA/dh)_0\}\cdot\{1/4M^2 - 0.8M - 0.2\}. \tag{23}$$

These relations can easily reduced to the results of Ishihara, Iwagaki and Y.Ishihara[17], if the flow is assumed two dimensional.

In turbulent flows, using Eq.(7) in the previous chapter, the instability criterion becomes

$$M(1 + b)u_{mo}/a(V_w - u_{mo}) \gtreqless 1, \qquad (24)$$

or

$$F_r^2 \gtreqless \{\lambda_o'' s_o/(dA/dh)_o\} \cdot [1/\{M^2(1 + b)^2/a^2 - 2(\beta_o - 1)M(1 + b)/a - (\beta_o - 1) - A_o(\partial\beta/\partial A)_o\}]. \qquad (25)$$

These are the initiation condition for hydraulic instability in turbulent flows, and it is evident the top sign implies instability and the lower stability. Especially, the left side of Eq.(24) is called as the Vedernikov number derived by Vedernikov[10] in 1945, and it means the supercritical flow in which the Vedernikov number is over unity is essentially unstable.

Recently, Iwagaki and the author[8] investigated the hydraulic characteristics of roll waves in steep chutes and derived the flow condition to sustain their final pattern of wave, based on the concept of Thomas[13] and Dressler[14]. The results are also the same expression as described here, and it may be quite interesting to be noted that the formation of roll waves in steep channels results from the hydraulic instability.

The present author[1] has already verified this criterion for the uniform flow, and the present equation of (25) is also an extended theorem involved the curvilinear influence and the dispersion effects of irregular distribution of momentum flux. As an example of such influences, the curvature of free surface will be introduced as the second approximation, and thus the flow characteristics like the frequency of disturbed wave will be investigated. With respect to this problem, a simple case of stability condition of two dimensional flow with uniform distribution of velocity will be concerned, owing to the great mathematical difficulty to treat with the general case.

The total pressure at a section is, from Eq.(36) in the previous chapter,

$$\int_0^h (p/\rho g)dy = (h^2\cos\theta/2) + (h^2u_m^2/3g)(\partial^2 h/\partial x^2) + (2h^2u_m/3g)(\partial^2 h/\partial x\partial t) + (h^2/3g)(\partial^2 h/\partial t^2) + (h^2/3g)(\partial u_m/\partial t)(\partial h/\partial x) - (u_m h/3g)(\partial h/\partial x)(\partial h/\partial t) - (hu_m^2/3g)(\partial h/\partial x)^2,$$

and the linearized version involving the curvature of free surface in the present simplified flow pattern is

$$(h_o/3)\{u_{mo}^2(\partial^4 h'/\partial x^4) + 2u_{mo}(\partial^4 h'/\partial x^3\partial t) + (\partial^4 h'/\partial x^2\partial t^2)\} + (g\cos\theta - u_{mo}^2/h_o)(\partial^2 h'/\partial x^2) - (2u_{mo}/h_o)(\partial^2 h'/\partial x\partial t) - (1/h_o)(\partial^2 h'/\partial t^2) - \{(u_{mo}/\rho h_o^2)(\partial\tau/\partial u)_o - (1/\rho h_o)(\partial\tau/\partial h)_o + (\tau_o/\rho h_o^2)\}(\partial h'/\partial x) - (1/\rho h_o^2)(\partial\tau/\partial u)_o(\partial h'/\partial t) = 0. \quad (26)$$

When a small disturbance added to the free surface is expressible by Eq.(18), Eq.(26) becomes a quadratic equation for γ, i.e.,

$$(1 + h_o^2L'^2/3)\gamma^2 + 2\{(1/2\rho h_o)(\partial\tau/\partial u_o) + iL'u_{mo}(1 + h_o^2L'^2/3)\}\gamma + L'^2(gh_o\cos\theta - u_{mo}^2) - (h_o^2u_{mo}^2L'^4/3) + (iL'/\rho)\cdot\{(u_{mo}/h_o)(\partial\tau/\partial u)_o - (\partial\tau/\partial h)_o + (\tau_o/h_o)\} = 0. \quad (27)$$

Finally, the instability condition for the two dimensional flow becomes, in terms of Froude number,

$$F_r^2 = 1/\{(1 + h_o^2L'^2/3)(1 + b)^2/a^2\}. \quad (28)$$

Comparing the above equation with Eq.(25) for the hydraulic instability, of great evidence is that the influence of frequency of a disturbance is introduced. If the wave length of disturbance is quite large compared with the flow depth, the instability condition is reduced to the Vedernikov criterion. On the contrary, for short wave length, the flow becomes unstable in low Froude number flow like subcritical flow. It indicates that the inherent property of open channel flow plays a primary role to the hydraulic instability. The present section deals with the mathematical description for hydraulic characteristics of flow stability, observed in rapid flows, by means of the usual one dimensional procedure of open channel flows. As have been indicated in several parts, there remain problems unsolved, to illustrate the true nature of stability of free surface in rapid flows. However, it seems for the author that the complete behaviour can not be theoretically and experimentally derived, if the dynamic equation is expressed in terms of the one dimensional procedure, and the experimental instruments have not been progressed with the advance of analytical work, though the basic characters of stability have been actually obtained by the present treatment.

1 - 2 - 4 General Theory of Transitional Characteristics of Steady Flows

The foregoing section concerned with the physical significance of two regimes of sub- and supercritical flows imposed by the free surface. The basic equation of steady flows in open channels is expressible by a relationship between the surface gradient and the non-linear term depending on the distance and the depth, as seen in Eqs.(1) - (3).

The critical depth, mathematically speaking, of steady flows in uniform channels is a singular point, and another singular point exists in the basic equation, if the numerator and the denominator become simultaneously zero. Nearly almost natural channels and even artificial watercourses in themselves involve local changes of channel geometry and boundary characteristics, which produce a variation in flow from one uniform state to another, and such transitions will be expected to make the basic equation in a form of 0/0. In fact, the mathematical expression of 0/0 yields the transition flows from tranquil to shooting or vice versa, as often seen in the later section. This type of problem was first studied by P. Massé[2] in

1938 in the case of flow over a variable slope as the further development of Bresse theory of quasi-linear flows. More recently, in 1956 F.F. Escoffier[3)] also investigated the same transitional behaviours of flows for the application to the graphical method of tracing of surface profiles of steady flows.

The purpose of this section is to reveal the basic properties of singular point being in a form of 0/0 and the hydraulic significance of transitional characteristics as a solution of basic equation of steady flows in the immediate vicinity of a singular point. The analysis expressed in this section may be considered as an extension of the classical theory of steady flows, which is characterized by the uniformity of channel geometry and developed for more than a century by a large number of scientists and engineers, and after the clarification of the present procedure of analysis the hydraulic characteristics of steady flows will also become of significance in the engineering application, which will be seen in the later part.

(a) Classification of Singular Point of Basic Steady Flows

The surface gradient equation of steady flows is represented by Eq.(1) for the energy approach. At a point where the numerator and the denominator become simultaneously zero, the surface gradient is indefinite. When the value at a critical singular point is expressed by the subscript c, the distance and the depth in the immediate vicinity of the singular point are expressible as

$$x = x_c + x', \text{ and } \quad h = h_c + h',$$

in which

$$x_c \gg x', \text{ and } \quad h_c \gg h'.$$

If the variation equation, which indicates the approximate behaviours of basic equation near the point, is derived, it is in a form of

$$(dh'/dx') = (cx' + dh' + Q)/(ax' + bh' + P) \tag{29}$$

in which, coefficients of a, b, c and d are definite numerical constants depending on the critical condition, and P and Q are higher terms of the squares and products of x' and h'.

Omitting the prime in Eq.(29), for the convenience of notation, Eq.(29) is the linear equation near the singular point and a solution will describe the approximate behaviours of steady flows and its mathematical properties will be obtained by means of the application of geometric theory of ordinary differential equation.

In the characteristic equation of Eq.(29),

$$S^2 - (a + d)S + (ad - bc) = 0, \tag{30}$$

let denote two roots by S_1 and S_2.

(1) When S_1 and S_2 are real and of opposite sign, a singular point is called saddle, and through the point two singular solutions of surface profiles pass.

(2) S_1 and S_2 are real and of same sign, so that a singular point is classified as a nodal point, at which all surface profiles of flows have a certain definite direction determined by the channel and flow characteristics in channel transitions and controls.

(3) If S_1 and S_2 are conjugate complex, a singular point becomes a focal point, and all surface profiles in the immediate vicinity of the point are logarithmic spirals and approach the point.

Denoting the discriminant of characteristic equation by D, the classification of singular points, therefore, as the transitional point of flows, is described in the following:

for saddle point, $D > 0$, and $ad - bc < 0$,

for nodal point, $D > 0$, and $ad - bc > 0$,

and for focal point, $D < 0$, and $ad - bc > 0$.

(b) Geometric Properties of Normal and Critical Depths Curves near Singular Point

When the numerator of basic equation becomes zero, the surface gradient also becomes zero, and the local uniform condition is obtained. The curve indicated by that the denominator is zero is the curve of critical depth for a particular discharge. Denoting the slopes of the foregoing two curves at the singular point by s_1 and s_2, they are derived by the linear variation equation of (29) as follows.

$$s_1 = -(c/d), \tag{31}$$

and

$$s_2 = -(a/b), \tag{32}$$

and therefore s_1 and s_2 are characterized by the channel and flow characteristics. The subtraction of s_2 from s_1 yields

$$s_1 - s_2 = (ad - bc)/bd. \tag{33}$$

If b and d are of same sign, the sign of $(s_1 - s_2)$ corresponds with that of (ad - bc), and if b and d are opposite, $(s_1 - s_2)$ and (ad - bc) are of opposite sign. Furthermore, the sign of (ad - bc) is negative for the saddle point and positive for the nodal and focal points as seen in the previous subsection.

On the other hand, when $(s_1 - s_2)$ is positive, the curve of normal depth passes through the lower half plane divided by the critical depth curve and approaches the singular point, despite of the sign of s_1 and s_2. When $(s_1 - s_2)$ is negative, the curve of normal depth passes from the upper plane to the lower plane through the singular point. Consequently, the relationship between the classification of singular points and the geometric properties of two curves is described as follows.

When b and d are of same sign, the normal flow curve passes from the upper plane to the lower plane divided by the critical depth curve through the saddle point, while the nodal and focal points induce the normal depth curve passes from lower to upper. On the contrary, when b and d are of opposite sign, the opposite behaviours against the foregoing conclusion are indicated.

The critical depth curve divides the whole plane into two parts of positive and negative signs. In the immediate vicinity of the singular point, the mathematical behaviours of critical depth curve is approximated by the straight line of $ax + bh = 0$. If b is positive, the upper half plane divided by the critical depth curve indicates the positive domain and the lower becomes negative, and on the contrary, the negative values of b makes the upper plane negative and the lower positive. In the same manner, the normal depth curve divides the whole plane into two domains and the upper domain becomes positive or negative depending on positive or negative signs of d.

As the conclusive description on the behaviours of normal and critical depths curves near the singular point, the followings are indicated.

(1) When a singular point is saddle, the normal depth curve passes the upper domain divided by the critical depth curve and approaches the saddle point for the same sign of b and d. The sign of surface gradient indicated by Eq.(1) is negative between both curves. For the case that b and d are of opposite sign, the normal depth curve passes the lower plane and the sign of surface slope is positive between two curves.

Furthermore in both cases, the sign of surface slope in the upper domain of critical depth curve is negative and this significant statement will lead the conclusion that the hydraulic behaviours of flow upstream from the saddle point are similar to those in the mild slope channel derived by the theory of Bresse and the downstream behaviours are also to those in the steep slopes, and it will be verified in the last part of this section.

(2) When a singular point is classified as nodal or focal, the opposite behaviours will be produced by the singular point, compared with the case (1). The upper domain divided by the critical depth curve makes the sign of surface profile positive.

Fig. 1-19 indicates the classification of signs of surface profiles at singular points for various cases of b and d.

(c) Surface Gradient of Transition Profiles of Flows as Solutions of Basic Equation at Singular Point

At the singular point as the transitional point, both numerator and denominator become simultaneously zero and the resulting surface slope derived by the original equation is indeterminate. When the singular point is classified as a channel control in the hydraulics of open channel flows, it is commonly a starting point for the calculation of surface profiles of water in design problems of hydraulic engineering, and consequently, the value of dh/dx at the point must be known for the calculation procedure.

The usual procedure to evaluate the value of surface gradient at the singular point is to use a method of form 0/0 in the differential calculus known as the rule of L. Hospital. Or simply, $(dh/dx)_{x,h\to o} = (h/x)_{x,h\to o} =$ cost., if these values at the singular point are certain definite, so that Eq.(29) becomes

$$b(dh/dx)^2 + (a - d)(dh/dx) - c = 0. \qquad (34)$$

The slope of transition profile is then

$$(dh/dx) = \{-(a - d) \pm \sqrt{(a - d)^2 + 4bc}\}/2b. \qquad (35)$$

The possible slope of surface profile is evidently one of solutions of Eq.(35). The rather empirical study of M. Homma[18] on the behaviours of surface profiles of Chézy flows in wide rectangular channels indicates that the possible slope is given by a negative

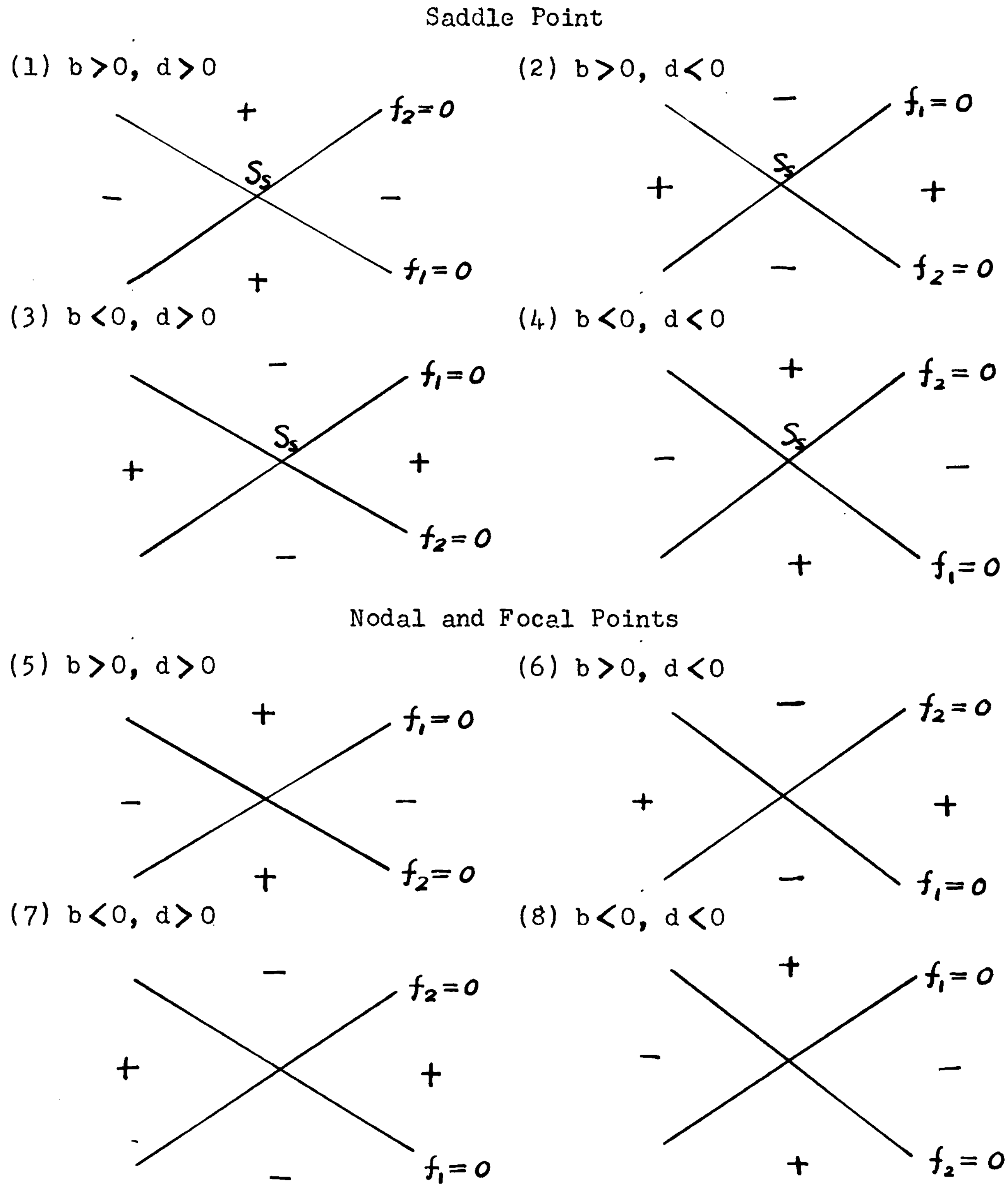

Fig. 1 -19 Classification of signs of surface profiles of water for various cases of b and d root of Eq.(35) for divergent channels and a positive one for convergent channels. As two roots are of same sign if c and b* are opposite, and therefore, the foregoing conclusion of Homma on the possible surface slope at the singular point is essentially insufficient to define the slope of open channel flows.

The first consideration to make the behaviours of slope and surface profiles clear is orientated to the flow passing through the saddle point. Denoting the positive root by S_2 and the negative one by S_1 in the characteristic equation, in which S_1 and S_2 are of opposite sign, the canonical form of the variation equation is derived with the use of the linear transformation being in forms of

$$x = (S_2 - a)\xi + (S_1 - a)\eta, \tag{36}$$

and

$$h = -c\xi - c\eta, \tag{37}$$

and it is

$$d\xi/(S_1\xi + \cdots\cdots) = d\eta/(S_2\eta + \cdots\cdots), \tag{38}$$

Consequently, the hydraulic behaviours of flow in the immediate vicinity of the singular point is approximately indicated by

$$d\eta/d\xi = -|S_2/S_1|(\eta/\xi). \tag{39}$$

The solution of Eq.(39) describes a family of hyperbolas, among which only two singular curves of surface profiles, C1- and C2-curves, pass through the saddle point, and the surface profile at the point is

$$(d\eta/d\xi)_c = 0, \infty.$$

Transforming to the original coordinate system, the surface gradient

---

* In the study of transitional behaviours of gradually varied flows, the positive value of b indicates the channel is divergent and the convergent channel is characterized by $b < 0$. The details are seen in the literature of the author[19].

is then

$$(dh/dx)_{c1} = - c/(S_2 - a), \tag{40}$$

and

$$(dh/dx)_{c2} = - c/(S_1 - a). \tag{41}$$

The expressions of Eqs.(40) and (41) are reduced to the same expressions of Eq.(35).

The surface gradient of transition flows at the nodal point will be next considered. The characteristic equation has two real roots of same sign. Let denote the greater root by S_2, and then S_2/S_1 is greater than unity. By means of the linear transformation of Eqs.(36) and (37), the canonical form becomes

$$\eta = C\xi^a, \qquad a = S_2/S_1 > 0.$$

The gradient of transition flows at the nodal point is then uniquely determined and it is

$$(dh/dx) = - c/(S_2 - a).$$

If S_2 is defined as the greater root for the nodal point, the mathematical form of surface gradient is the same expression as that for C1-curve.

The focal point makes the surface profiles logarithmic spirals near the singular point and no definite direction is produced. Such surface profiles can not be possible in the open channel flows and consequently the hydraulic jump will occur, when the up- and downstream depths of flow become conjugate in the momentum conservation law in channels, and the flow changes suddenly its flow regime.

(d) Transitional Behaviours of Flow Through Singular Point

The most significant subject of the study on the transitional behaviours of steady flows and associated problems in channel design of conveyance and control structures is to formulate the type of change in flow regime for particular design discharge through the singular point. When a singular curve as a surface profile approaches the singular point after passing through the upper half plane divided by the critical depth curve, and thereafter passes through the lower plane, the flow changes its regime from tranquil to shooting. On the contrary, the singular curve passes from the lower plane to the upper plane, so that the flow regime changes from shooting to tranquil. Consequently, the change of flow regime from tranquil to shooting or vice versa is solved by the establishment of geometrical properties between the singular curve and the critical depth curve for the design discharge.

(1) Transitional behaviours through saddle point

The case, in which the singular point is classified as a saddle point and the singular slope is assumed to be $(dh/dx)_{c1}$, will be first treated.

Making the subtraction of s_2 from $(dh/dx)_{c1}$ yields

$$(dh/dx)_{c1} - s_2 = (S_1/b). \tag{42}$$

As S_1 is negative, so the geometric properties between C1-curve as one of the singular solutions and the critical depth curve are determined by the sign of b. For the positive value of b, C1-curve crosses the saddle point from the upper plane to lower plane of the critical depth curve. On the contrary, for the negative value of b, C1-curve passes through the point from lower to upper.

In the same manner,

$$(dh/dx)_{c2} - s_2 = (S_2/b), \tag{43}$$

so that C2-curve passes through the saddle point from lower to upper or from upper to lower, depending on positive or negative values of b.

The difference between two values of slopes of C1- and C2-curves is

$$(dh/dx)_{c1} - (dh/dx)_{c2} = -(D/b). \tag{44}$$

As the discriminant is positive, so C1-curve exists in the upper domain of C2-curve and intersects together at the saddle point for $b > 0$. For $b < 0$, the opposite behaviour is indicated in x-h plane.

Finally, for the positive value of b, C1-curve passes the upper plane of the critical depth curve, and C2-curve the lower plane, while C1-curve passes the lower plane of the critical depth curve and C2-curve is in the upper plane for $b < 0$.

In the subcritical flow, on the other hand, the surface profile must be proceeded from the downstream end as already indicated, and furthermore, the surface slope of flow traced under a particular boundary condition is negative in the vicinity of singular point, and therefore the tranquil branch of singular solutions becomes the asymptote in the upstream region from the saddle point. Despite of the sign in b, consequently, of great significance in the transitional behaviour through the saddle point is that the flow changes its regime from tranquil to shooting and the transition profile of flow is expressed by C1-curve for the positive value of b and by C2-curve for $b < 0$.

(2) Transitional behaviours through nodal point

In the same manner as did for the study of transitional behaviours through the saddle point, the geometric property between the transitional slope of flow and that of the critical depth curve will be treated. As

$$(dh/dx)_c - s_2 = (S_1/b). \tag{45}$$

so, when S_1 and b are of same sign, the transitional behaviours indicates that the flow changes its regime from shooting to tranquil through the nodal point and the nodal point is a terminal for the calculation procedure in surface profile routing. On the contrary, when S_1 and b are opposite, it is seen the nodal point becomes the starting point for the calculation and furthermore the flow changes from tranquil to shooting, bearing in mind that the subcritical flow must be proceeded from the downstream end. All curves of surface profiles, however, as solutions of the basic equation have the same slope at the nodal point, and if the nodal point is a starting point, the surface profile can not be determined. On the other hand, if the point becomes a terminal, the surface profile traced under the given condition can be uniquely determined. The nodal point, therefore, will be the terminal for the calculation procedure of surface profile and the flow changes from shooting to tranquil. It is consequently expected that S_1 and b are not opposite in this case. The details of description for the flow with particular resistance laws will be presented in the later parts of this study and actually, in almost all cases, the nodal point can not be the transitional point and the flow changes rapidly from shooting to tranquil by the hydraulic jump in the neighbourhood of nodal points.

(3) Transitional behaviours through focal point

As already indicated in the foregoing subsection, the surface profiles near the focal point are logarithmic spirals and no definite surface slopes in direction and magnitude are indicated at a section. In the actual open channel flows, such transitional behaviours can not be produced. Only the way possible to change the flow regimes is due to the hydraulic jump and therefore the flow changes from shooting to tranquil at a section where the up- and downstream depths traced by other boundary conditions imposed by control structures and the like become sequent in the momentum conservation law.

1 - 2 - 5 Verification of Jaeger's Generalized Theory on Simultaneity of Maximum Discharge and Minimum Total Head in Connection with Bélanger-Böss Theorems

As has been described so far, the critical depth h_c in the hydraulics of steady flows was introduced in three ways:

(1) when the surface slope in the basic equation becomes quite large to infinity, and the flow changes from tranquil to shooting or vice versa,

(2) when the discharge becomes a maximum value for a given total head.

(3) or when the total head becomes minimum for a particular discharge.

The first way is known as the definition of critical depth in the theory of steady flow hydraulics developed since Bresse during the 19 th and 20 th centuries. The second is the well known theorem of Bélanger[20] who used it for the calculation of discharge over a broad crested weir in 1849, and also was treated by Boussinesq[21] in 1877, without establishing the theorem by rigorous proof. The last one is proved by P. Böss[4] in 1919, who observed the flow in transition from tranquil to shooting passed through a critical depth when the total energy line was at its lowest level.

Furthermore, the critical depth is also introduced on the basis of unsteadyness of open channel flows, as already described, which provided possibly the essential character of the calculation procedure of surface profiles. The important problem is then to establish the significant interrelationship among three ways of definition.

Confined the problem only to the steady flow, the first description of the simultaneous occurrence of maximum discharge and minimum total energy is referred to the literature of de Marchi[22] in 1930, if the stream line in the flow is straight, and in fact, the numerical values of h_c in the case of parallel streamlined flows can easily be proved to be identical for all three conditions above indicated. The foregoing indications described as well as all opinions of a large number of scientists and engineers, despite of the lack of definite mathematical proof of the theorem, are that the concept of critical depth may be extrapolated to the curved flow. In 1943, C. Jaeger[23)] treated generally with the simultaneity theorem of maximum discharge and minimum total energy for the open channel flow involving both parallel and curved flows, and the theorem derived is known as the Jaeger's general theory or the generalized Bélanger-Böss theorems. With the use of the specific energy in the energy approach, he obtained the simultaneous occurrence of $(\partial Q/\partial h) = 0$ and $(\partial H_0/\partial h) = 0$ at a transitional point. His intention will be supposed to orientate to the clarification of the one dimensional transitional behaviour, which is the main purpose of the study, nevertheless, he had no definite idea on the behaviour of transition from tranquil to shooting, so that his theory is of self-evidence in the basic physics of open channel flows, based on the one dimensional method of energy approach.

Mathematical proof on the basic philosophy of Jaeger's general theory and its hydraulic significance will be presented in simple but evident forms, when the transitional theory of open channel flows through the saddle point described in the previous section is introduced.

The energy approach as a mean for the analysis of hydraulic behaviours of open channel flows will be used to establish the interrelationship between Bélanger and Böss theorems.

As indicated in 1-2-2, the surface gradient of steady flows is in a form of

$$(dh/dx) = \{\sin\theta - (\tau/\rho gR)(u_b/u_m) - (\partial H_o/\partial x)\}/(\partial H_o/\partial h). \quad (46)$$

If the flow under consideration possesses a saddle point as a transitional point in its flow regime, the numerator and denominator become simultaneously zero, i.e.

$$(\partial H_o/\partial h) = 0. \quad (47)$$

It follows the minimum total energy is obtained at the saddle point for a particular discharge Q, and furthermore the saddle point indicates a transition from tranquil to shooting, so that the theorem of Böss, which provides the minimum total energy at the transitional point from tranquil to shooting, is established.

The specific energy H_o is defined as, from Eq.(67) in the previous chapter,

$$H_o = (1/Q)\int\{(u^2 + v^2)/2g + p/\rho g + y\cos\theta\}udA,$$

and thus the specific energy, discharge, other flow and channel characteristics and so on are connected in an implicite form of

$$F(H_o, Q, h, R, \theta, \cdots\cdots) = 0. \quad (48)$$

Under the condition of constant discharge,

$$(\partial F/\partial H_o)(\partial H_o/\partial h) + (\partial F/\partial h) + (\partial F/\partial R)(\partial R/\partial H_o)(\partial H_o/\partial h) + (\partial F/\partial R)(\partial R/\partial h) + \cdots\cdots = 0,$$

or

$$(\partial H_o/\partial h) = -\{(\partial F/\partial h) + (\partial F/\partial R)(\partial R/\partial h) + \cdots\cdots\}/\{(\partial F/\partial H_o) + (\partial F/\partial R)(\partial R/\partial H_o) + \cdots\cdots\}. \quad (49)$$

Similarly, for constant specific head on a channel bottom of all slopes,

$$(\partial F/\partial Q)(\partial Q/\partial h) + (\partial F/\partial h) + (\partial F/\partial R)(\partial R/\partial Q)(\partial Q/\partial h) + (\partial F/\partial R)(\partial R/\partial h) + \cdots\cdots = 0,$$

or

$$(\partial Q/\partial h) = -\{(\partial F/\partial h) + (\partial F/\partial R)(\partial R/\partial h) + \cdots\cdot\}/\{(\partial F/\partial Q) + (\partial F/\partial R)(\partial R/\partial H_0) + \cdots\cdot\}. \tag{50}$$

Neither $(\partial F/\partial H_0) + \cdots$ nor $(\partial F/\partial Q) + \cdots$ are infinite, when the flow passes through the channel, so that $(\partial H_0/\partial h) = 0$ makes $(\partial F/\partial h) + (\partial F/\partial R)(\partial R/\partial h) + \cdots\cdot$ zero, and $(\partial Q/\partial h) = 0$ at the saddle point.

The theorem of simultaneity of maximum discharge and minimum total energy known as the generalized Bélanger-Böss theorems or the generalized Jaeger's theory, therefore, is verified by means of the geometric theory of transitional behaviours of open channel flows.

The most significant and important implication of this theorem is that it provides an essential comprehension of the hydraulics of open channel flows to the analytical methods of transitional behaviours by control structures of a wide variety in hydraulic functions. As a matter of fact, in the discharge measurement by control structures, this relationship is commonly applied, without any detailed knowledge on the transitional behaviours of flow. For example, Bélanger and Boussinesq used it for the calculation of discharge over a broad crested weir, Boussinesq for the flow over a sharp crested weir and Jaeger[24)] for the flow over a round crested weir. Among them, the most effective application of this theorem to the flow measurement is seen in the standing wave flume, the Parshall flume and the like, in which the flow is classified as a typical example of gradually varied flows, owing to that the basic physics of flow can be mathematically expressed in a more exact form than that in rapidly varied flows through weirs. These detailed characteristics in hydraulics behaviours will be evidently explained in the later parts.

The theorem between the maximum discharge and minimum momentum flux in a uniform channel and the possibility for the establishment of connection between energy and momentum flux approaches will be next treated.

Under the condition of steady flows in uniform channels, the surface profile equation by the momentum approach is

$$(dh/dx) = \{A\sin\theta - (\tau s/\rho g) - (\partial M_0/\partial x)\}/(\partial M_0/\partial h), \qquad (51)$$

so that, in the same manner as did in the foregoing case, the theorem of simultaneity of maximum discharge and minimum momentum flux is established and the momentum theorem may possess properties analogous to those of the energy theorem. Herewith, one important but deeply difficult problem of the connection between both approaches of energy and momentum flux in open channel flows known as the stability of flow presented by Boussinesq will be arised. As already indicated in the previous chapter, it is commonly seen both expressions by momentum and energy approaches can not be coincided. This is due to the difference in the dynamic treatment of basic principles between the energy approach as a scalar equation and the momentum approach as a vector equation and to the great difficulty and complexity to express the complete feature of the physics of flow in an exact mathematical form as indicated by Boussinesq and Jaeger, and the perfect connection between both theorems will be obtained after the real establishment in the clarification of basic physics of flow is furnished.

---

1) Chow, V.T., Integrating the Equation of Gradually Varied Flow, Proc. ASCE, Separate 838, Nov. 1955.

2) Massé, P., Ressaut et ligne d'eau dans les cours d'eau ā pente variable, Rev. gen. Hydraulique, Nos. 19-20, Jan.-April 1938.

3) Escoffier, F.F., Transition Profiles in Non-Uniform Channels, Jour. Hydraulics Division, Proc. ASCE, HY 3, 1956.

4) Böss, P., Berechnung der Wasserspiegellage beim Wechsel des Fliesszustandes, Springer, Berlin, 1919.

5) Iwasa, Y., The Criterion for Instability of Steady Uniform Flows in Open Channels, Memoirs , Fac. of Eng., Kyoto University, Vol. 16, No. 4, Oct. 1954.

6) Jeffreys, H., The Flow of Water in an Inclined Channel of Rectangular Section, Phil. Mag., Series 6, Vol. 49, 1925.

7) Keulegan, G.H., and Patterson, G.W., A Criterion for Instability of Flow in Steep Channels, Trans. AGU, Part II, 1940.

8) Iwagaki, Y., and Iwasa, Y., Hydraulic Characteristics of Roll-Wave Trains, Proc. JSCE, Vol. 40, No. 1, Jan. 1955 (in Japanese).

9) Cornish, V., Waves of the Sea and Other Water Waves, Fisher Unwin, London, 1910.

10) Vedernikov, V.V., Conditions at the Front of a Translation Waves Disturbing a Steady Motion of a Real Fluid, (Comptes Rendus), Doklady Akademia Nauk, SSSR, Vol. 48, 1945.

11) Craya, A., The Criterion for the Possibility of Roll Wave Formation, Proc. Gravity Wave Symposium, NBS, 1951.

12) Dressler, R.F., and Pohle, F.V., Resistance Effects on Hydraulic Instability, Comm. Pure and Appl. Math., Vol. 6, No. 1, 1953.

13) Thomas, H., The Propagation of Waves in Steep Prismatic Conduits, Proc. Hyd. Conference, University of Iowa Studies in Eng., Bull. 20, 1940.

14) Dressler, R.F., Mathematical Solution of the Problem of Roll Waves in Inclined Open Channels, Comm. Pure and Appl. Math., Vol. 2, No. 2-3, 1949.

15) Ishihara, T., Iwagaki, Y., and Iwasa, Y., Theory of the Roll-Wave Trains in Laminar Water Flow on a Steep Slope Surface, Trans. JSCE, Vol. 19, 1954 (in Japanese).

16) Ishihara, T., Iwagaki, Y., and Goda, T., Studies on the Thin Sheet Flow, Trans. JSCE, No. 6, Aug. 1951 (in Japanese).

17) Ishihara, T., Iwagaki, Y., and Ishihara, Y., Studies on the Thin Sheet Flow, 3rd Report, Rain Wave Trains, Proc. JSCE, Vol. 36, No. 1, 1951 (in Japanese).

18) Homma, M., Hydraulics (Fluid Mechanics for Hydraulic Engineers), Maruzen, Tokyo, 1952 (in Japanese).

19) Iwasa, Y., Theoretical Study of Hydraulic Behaviours of Boundary Characteristics to Channel Transitions and Controls in Divergent and Convergent Channels, Trans. JSCE, No. 59, Separate 3-1, Nov. 1958 (in Japanese).

20) Bélanger, J.B.Ch., Notes sur le cours d'hydraulique, Mém. École nat. Ponts Chaussées, Paris, 1849-50.

21) Boussinesq, J.V., Essai sur la théorie des eaux courantes,

Mém. présentés par divers savants à l'Académie des Sciences, Paris, 1877.

22) de Marchi, G., Idraulica, 1930.

23) Jaeger, C., Contribution à l'étude des courants liquides a surface libre, Rev. gen. Hydraulique, Nos. 33 and 34, 1934.

24) Jaeger, C., Remarques sur quelques écoulements le long de lits à pente variant graduellement, Schweiz. Bauztg., Vol. 114, No. 20, 1939.

3. Transient Characteristics of Open Channel Flows

2 - 3 - 1 General Scope of Problems in Conveyance Channels

In the hydraulic design of conveyance canals, the establishment of transient characteristics of open channel flows is often necessary. When the power canal is designed, the height of side wall necessary to prevent overflows during load rejection is carefully estimated and before the construction of navigation canals the allowable velocity for the influence to shipping and the minimum depth of water to provide the adequate draft for vessels are required. It is, however, quite difficult to clarify all various forms of unsteady phenomena of flows in open channels, which have their own special peculiarities. The most familiar classification of transient flows to hydraulic engineers is in the following.

(1) flood waves in natural channels;

(2) translation waves in navigation canals by locking operation;

(3) surges in open channels like power and irrigation canals, resulting from the sudden increase or decrease in load.

All types of flow are translation waves in which the particle velocity is the same, or nearly so, at all points of any section, as defined by G.H. Keulegan[1)] and others. On the other hand, the particle motion in sea waves is practically ignored compared with the progression of surface configuration.

A striking distinction, however, in the character of surface profiles among the above three types of transient flows will be comprehended. In cases of (1) and (2), the vertical component of the acceleration of fluid flows, compared with the total acceleration, is small and the resulting change in surface profiles is so gradual that the boundary resistance becomes of great importance to the transient behaviours. On the other hand, in the third case, the instantaneous change in flow characteristics is quite large and thus the resistance plays a secondary sequence to the effect for the basic equation.

Another example of transient problems in open channels is the intermittent surge known as roll waves, which are observed in steep chutes and especially in the thin sheet flows[2]. Detailed knowledge of roll wave characteristics is essentially needed for the hydraulic design of steep conveyance structures.

In this chapter, the basic transient characters of open channel flows necessary to the hydraulics of conveyance of water are concerned. This character has been studied by many hydraulic engineers and also a large number of publications[3], [4], [5] have been presented up to the present time. For this reason, the present chapter is restricted to describe some additional contributions to design problems.

The first section concerns with the approximate behaviours of translation wave and surges by means of the concept of a discontinuity shock condition for various cases of channel geometry. This phenomenon has been often treated and fruitful results also are derived in various forms. Especially, the special case in which the absolute velocity of ascending wave becomes zero is known as the hydraulic jump. The second will treat with the influence of vertical component of particle acceleration in the flow. The resulting surface configuration becomes cnoidal or solitary as observed by Boussinesq and Keulegan in the higher order solution. As the solitary wave is a stable form of appreciable wave height resulting from the considerable magnitude of vertical acceleration, though the original analysis has been based on the mathematical interests, so the behaviour of translation wave will be much contributed by the solitary wave analysis.

1 - 3 - 2 First Order Behaviours of Surges and Translatory Waves

When the sudden increase of discharge in the canal is released, the surge defined as a long wave, in which the effects of surface curvature are negligible compared with those of boundary resistance, and the translation wave are resulted, being in forms of discontinuous shock front.

In connection with the propagation of surges and translation waves the classical theory is essentially based on the linearized characteristics of unsteady open channel flows. As one of such analyses, P. Massé[6] obtained a fruitful result for the velocity propagation and the wave characteristics. The non-linear theory of velocity propagation in unsteady flows gradually developed by a large number of engineers provides the basic principles of the flood routing procedures in open channels, and freqqently is used by engineers to find the general characters of flood waves in channels.

Recent trend for practical calculation of velocity of surges initiated by J. Massau[7] as long ago as 1889 is to use the so-called method of characteristics furnished with the theory of partial differential equation. R. Ré[8] calculated the propagation velocity of discontinuous surge front known as the dam break function in 1946, this procedure again becomes to be widely used to various problems in hydraulic engineering, and furthermore with the development of digital computer, many varieties of application of this procedure to hydraulic projects become possible. The flood routing problem[9] at the junction of the upper Mississippi River and the Ohio River is the most famous example of the application of method of characteristics to the transient characters of open channels by means of the digital computer.

Aside from the recent trend to the transient problems which can be solved by means of the method of characteristics, the propagation of surges and translation waves, from the classical point view of hydraulics, may be obtained by Eq.(120) in 1-1-6, as a primary approximation, which is

$$V_w = u_o \pm \sqrt{\{A_1/\rho A_o(A_1 - A_o)\}(\int_{a_1} pdA - \int_{a_o} pdA)} \ .$$

At values of (h_1/h_o) greater than 2, the front has usually the form of a breaking wave, the length of the breaker being approximately 5 times the depth h_1, and the stationary one is called the hydraulic jump of normal type. Under this condition, the pressure in the moving fluid is essentially hydrostatic, and then the velocity of propagation is

$$V_w = u_o \pm \sqrt{gy_{Go}\cos\theta\{(A_1 y_{G1}/A_o y_{Go}) - 1\}/(1 - A_o/A_1)} \ .$$

At values of (h_1/h_o) less than 2, but greater than 1, the depth change is not abrupt but occurs as a series of surface undulation about the depth h_1. The translation wave is probably of cnoidal form, which will be treated in the next section.

Surges produced in canals by the sudden changes in load at the power station are typical examples of this phenomenon. The hydraulic characteristics of surges and translation waves are studied by many hydraulic engineers, and especially H. Favre[5)], J. Frank[3)], G.R. Rich[4)] made the complete investigation of the behaviours. Favre also obtained the calculation procedure for surges travelling upstream, which are characterized by the appreciable influence of boundary resistance.

For hydraulic designs of canals and conveyance structures, one of the most important problems is the transformation of wave characteristics resulted from the channel transitions. The abrupt discontinuities in channels like positive and negative steps and the local changes in cross section commonly known as channel transitions divide the incident wave into a transmitted and a reflected wave. With the use of two relationships of continuity and wave height for incident wave and transmitted wave as well as reflected one, Frank[3] classified the flow behaviours of translation waves for various cases in channel transitions, some of which are seen in studies of Krey and Forchheimer. For positive and negative steps, the absolute velocity which can be obtained by the momentum conservation law described in 1-1-6 is transformed by the reflection of pressure. In the steady flow, the upstream face of positive step is acted by the hydrostatic pressure as H.A. Doeringsfeld and C.L. Barker[10] found experimentally during their studies on the hydraulic performance of broad crested weir. If the same assumption for the pressure distribution in transient flows as in the steady flow will be used, the absolute velocity may be calculated, and therefore the transformation of incident translation waves can also estimated as a first approximation.

1 - 3 - 3 Hydraulic Characteristics of Translation Waves of Finite Amplitude

The preceding section concerned with the propagation of rapid change in water elevation. The wave front is generally considered breaking. In this section, the propagation of translation waves of finite amplitude and their hydraulic characteristics will be treated, as the second order theory of waves in open channel.

Translation waves are essentially unsteady in the category of gravity waves, and the particular feature of this type of waves in open channels is indicated by the permanent displacement of fluid particles. The assumption of constancy in absolute velocity of propagation, however, makes the analysis simple because of eliminating of time derivatives. The waves of this type are divided into two ways: one is the case in which the frictional resistance becomes of great significance to the movement of water flows and the flood wave is a typical example, and the other is common procedure of analysis under the condition of equilibrium between the gravity force and the bottom resistance. The definition of translation wave is usually subjected to the latter type of waves of a permanent type.

The first approximation treated with small waves is known as the long wave theory. The pressure is substantially hydrostatic. In the second order theory, translation waves are characterized by the appreciable magnitude of vertical acceleration. The problems have been studied for many years by a large number of scientists and engineers. The first procedure of analysis which is a common method of analysis, is that the irrotationality of fluid flow is assumed. Boussinesq, Rayleigh, and more recently, G.H. Keulegan and G.W. Patterson[11)], T.B. Benjamin and M.J. Lighthill[12)], B.A. Packam[13)] and others investigated the wave characteristics by this approach. The other is to derive the second approximation of flow behaviours as waves of open channels, as Sérre[14)] and the author[15)] did. The hydraulic and mathematical expression of the first order theory is indicated by the sinusoidal wave, whereas the second order theory by the cnoidal waves in terms of the elliptic function of Jacobi.

The solitary wave as the limiting case of cnoidal wave has an important influence to hydraulics of open channel flows, as it is of stable form and can travel without changes in its surface profile, so that surges of appreciable height in channels are frequently transformed into the solitary wave or similar types of waves. The wave pattern is not readily changed and the hydraulic design of side walls must also take the consideration of this influence.

When the constancy of propagation velocity is assumed as an engineering approximation, the translation wave is easily analyzed in a simple form by means of the theory of moving discontinuity, while the usual procedure is based on the unsteady equation of fluid flows. The first subsection deals with the basic relationship of velocity of propagation and is followed by the hydraulic characteristics of cnoidal and solitary waves as the solution of second order theory in other subsections. The results derived by many engineers are almost the same, whereas the details in characters are different from others owing to the distinction of basic assumption and mathematical procedures. Nevertheless, the wave pattern in the second order theory of fluid flow is expressed in forms of cnoidal and solitary waves.

(a) Second Order Solution of Translation Waves by Means of Discontinuity Theory

In 1-1-6, the momentum and mass conservation laws for the flow involving a discontinuity were treated. If the pressure is non-hydrostatic and characterized by Eq.(36) of 1-1-3, the momentum conservation law of incoming waves over original flows becomes, denoting the hydrostatic flow by the subscript o,

$$\begin{aligned}\rho uh(V_w - u) - \rho u_o h_o(V_w - u_o) &= (\rho g\cos\theta h^2)/2 + (\rho h^2u^2/3)(\partial^2 h/\partial x^2)\\ &+ (2\rho h^2 u/3)(\partial^2 h/\partial x\partial t) + (\rho h^2/3)(\partial^2 h/\partial t^2) + (\rho h^2/3)(\partial u/\partial t)(\partial h/\partial x)\\ &- (\rho hu/3)(\partial h/\partial x)(\partial h/\partial t) - (\rho hu^2/3)(\partial h/\partial x)^2 - (\rho g\cos\theta h_o^2/2),\end{aligned} \tag{1}$$

for the two dimensional flow of uniform velocity.

Transformation of the origin to the moving one with constant velocity, $x - V_w t \rightarrow x$, can eliminate the time derivatives in the above equation and it is

$$uh(V_w - u) - u_o h_o(V_w - u_o) = g\cos\theta(h^2 - h_o^2)/2 + (h^2/3)(V_w - u)^2$$

$(d^2h/dx^2) - (h^2V_w/3)(du/dx)(dh/dx) + (huV_w/3)(dh/dx)^2 - (hu^2/3)(dh/dx)^2.$

Eliminating u from the above equation with the use of mass conservation law, the following relationship for the propagation velocity is obtained.

$$(V_w - u_o)^2(h_o/h)(h - h_o)\{1 + (h_o/3)(h - h_o)^{-1}(dh/dx)^2 - (hh_o/3)(h - h_o)^{-1}(d^2h/dx^2)\} = g\cos\theta(h^2 - h_o^2)/2. \qquad (2)$$

Solving for V_w, the absolute velocity of wave is

$$V_w = u_o \pm \sqrt{gh_o\cos\theta}(1 + \eta/h_o)^{1/2}(1 + \eta/2h_o)^{1/2}\{1 + (h_o/3\eta)(d\eta/dx)^2 - (h_o^2/3\eta)(1 + \eta/h_o)(d^2\eta/dx^2)\}^{-1/2}, \qquad (3)$$

in which $h - h_o = \eta$. This is the expression of absolute velocity for translation waves of finite amplitude, and evidently Eq.(3) is reduced to the well known formula of surges, if the influence of surface curvature and slope is ignored. With respect to this relationship, the Boussinesq-Keulegan-Patterson formula is largely familiar to hydraulic engineers. Expanding Eq.(3) by means of the theorem of Taylor, the lowest term of Eq.(3) becomes

$$V_w = u_o \pm \sqrt{gh_o\cos\theta}\{1 + (3\eta/4h_o) + (h_o^2/6\eta)(d^2\eta/dx^2)\}, \qquad (4)$$

and it is the same relationship of Boussinesq-Keulegan-Patterson, so that Eq.(4) is also an approximation of the second order theory of finite amplitude waves. This fact is easily verified by comparison of propagation velocity between the wave of Eq.(4), $V_w = u_o \pm \sqrt{gh_o\cos\theta}\{1 + (\eta/2h_o)\}$, and that derived from the basic equation of (1), $V_w = u_o \pm \sqrt{g(h_o + \eta)\cos\theta}$, though it is seen that the wave pattern of Eq.(4) is cnoidal or solitary, as Eq.(4) is transformed into $(d\eta/dx)^2 = f(\eta^2, \eta^3)$.

(b) Cnoidal Waves as the Solution of Basic Relationship

The problem is essentially unsteady, so that the analysis must be solved by the basic equation of unsteady open channel flows. In the same manner as did in the preceding subsection, the progressive wave flow is treated by the moving coordinate system. Limiting the wave flow only in the case of descending one, the momentum equation (71) in 1-1-4 becomes

$$(K^2/3)(d/dx)\{(d^2h/dx^2) - (1/h)(dh/dx)^2\} + (gh\cos\theta - K^2/h^2)(dh/dx) = gh\sin\theta - (\tau/\rho), \tag{5}$$

in which K is the progressive discharge rate of $(V_w - u)h$. integrating Eq.(5) with respect to x, putting the condition that (dh/dx) and (d^2h/dx^2) are not necessarily vanished at $h = h_1$, the result becomes

$$(d^2h/dx^2) - (1/h)(dh/dx)^2 = (3C_1/2K^2) - (3/h) - (3g\cos h^2/2K^2) + (3/K^2)\int(gh\sin\theta - \tau/\rho)dx, \tag{6}$$

in which C_1 is a constant. When the equilibrium state between the gravity force and the resistance is considered, the last term of the above integro-differential equation is ignored, so that Eq.(6) is again integrable and it is

$$(dh/dx)^2 = -(3g\cos\theta/K^2)\{h^3 - (C_2K^2/g\cos\theta)h^2 + (C_1/g\cos\theta)h - (K^2/g\cos\theta)\}, \tag{7}$$

where C_2 is another constant of integration.

If the energy procedure is applied to this problem, the similar result is obtained in the following.

$$(dh/dx)^2 = -(3g\cos\theta/K^2)\{h^3 - (C_1'/g\cos\theta)h^2 - (C_2'K^2/g\cos\theta)h - (K^2/g\cos\theta)\}, \tag{8}$$

in which C_1' and C_2' are constants.

In both cases, the polynomials of Eqs.(7) and (8) have three real positive roots, say h_{max}, h_{min}, and $K^2/gh_{max}h_{min}\cos\theta$. Consequently, the surface profile equation of open channel flows with the appreciable amount of vertical acceleration is

$$(dh/dx)^2 = (3g\cos\theta/K^2)(h_{max} - h)(h - h_{min})(h - K^2/gh_{max}h_{min}\cos\theta). \qquad (9)$$

Introducing new variables, $h = h_{max}\cos^2\chi + h_{min}\sin^2\chi$, and selecting the origin at the apex, $h = h_{max}$ and $x = 0$, the surface profiles is expressed by the elliptic integral of

$$x = \Delta\int_0^{\chi}(1/\sqrt{1 - k^2\sin^2\chi})\,d\chi = \Delta F(\chi, k), \qquad (10)$$

in which $F(\chi, k)$ is an incomplete elliptic integral of the first kind,

$$\Delta = 2K/\sqrt{3g\cos\theta(h_{max} - K^2/gh_{max}h_{min}\cos\theta)},$$

and

$$k^2 = (h_{max} - h_{min})/(h_{max} - K^2/gh_{max}h_{min}\cos\theta) \leq 1.$$

With the use of the Jacobi's elliptic function, since $\mathrm{sn}\, x = \sin\chi$, $\mathrm{cn}\, x = \cos\chi$, and $\mathrm{sn}^2 x + \mathrm{cn}^2 x = 1$. then Eq.(10) becomes

$$h = h_{min} + (h_{max} - h_{min})\mathrm{cn}^2(x/\Delta, k), \qquad (11)$$

or reverting to the original coordinate system,

$$h = h_{min} + (h_{max} - h_{min})\mathrm{cn}^2\{\sqrt{(3g\cos\theta/4K^2)(h_{max} - K^2/gh_{max}h_{min}\cos\theta)}\cdot(x - V_w t), k\}. \qquad (12)$$

This equation represents an infinite number of undulations of identical shape, each symmetrical about a vertical plane passing through the apex, and the waves moving without changes in their forms are called as cnoidal waves being derived therefrom as analogous to sinusoidal by Korteweg and deVries.

If the variations from the undisturbed level are used as Keulegan and Patterson did, $h = h_o + h'$, $h_{max} = h_o + h'_{max}$, $h_{min} = h_o - h'_{min}$ and $K^2/gh_{max}h_{min}\cos\theta = h_o - h'_3$, and then Eq.(12) becomes

$$h' = -h'_{min} + (h'_{max} + h'_{min})\mathrm{cn}^2\{\sqrt{(3/4)(h'_{max} + h'_3)(h_o + h'_{max})^{-1}(h_o - h'_{min})^{-1}(h_o - h'_3)^{-1}}\cdot(x - V_w t), \sqrt{(h'_{max} + h'_{min})/(h'_{max} +}$$

$\overline{h_3'})$. (13)

When the higher terms are ignored, Eq.(13) is

$$h' = -h'_{min} + (h'_{max} + h'_{min})cn^2\left\{\sqrt{3(h'_{max} + h'_3)/4h_o^3}\cdot(x - V_w t),\ \sqrt{(h'_{max} + h'_{min})/(h'_{max} + h'_3)}\right\}, \tag{14}$$

and reduced to the same result of Keulegan and Patterson. Consequently, the analysis of Keulegan and Patterson is more approximate than that derived here.

Concerning the hydraulic characteristics of cnoidal waves, the following summaries[15)] are obtained, by the foregoing discussion.

(1) The flow regime at the apex is commonly tranquil, whereas that at the trough can not be determined and is subcritical or supercritical.

(2) The wave length L of cnoidal waves is

$$L = 2\Delta F_1(\pi/2,\ k), \tag{15}$$

in which $F_1(\pi/2, k)$ is the complete elliptic integral of the first kind.

(3) The propagation velocity is calculated by the try and cut procedure, with the use of

$$gh_{max}h_{min}\cos\theta\{h_o F_1(\pi/2,\ k) - h_{max}E_1(\pi/2,\ k)\} = K^2\{F_1(\pi/2,\ k) - E_1(\pi/2,\ k)\}, \tag{16}$$

in which $E_1(\pi/2, k)$ is the complete elliptic integral of the second kind, if the wave height is known.

Fig. 1-20 indicates the dimensionless wave celerity $c' = c/\sqrt{gh_o\cos\theta}$ as a function of dimensionless wave length $L' = L/h_o$. The dispersive property of cnoidal waves is apparently seen and the celerity approaches that of solitary waves, if the wave length becomes larger. On the other hand, if the wave length becomes shorter, the cnoidal wave becomes similar to the sinusoidal wave, as will be

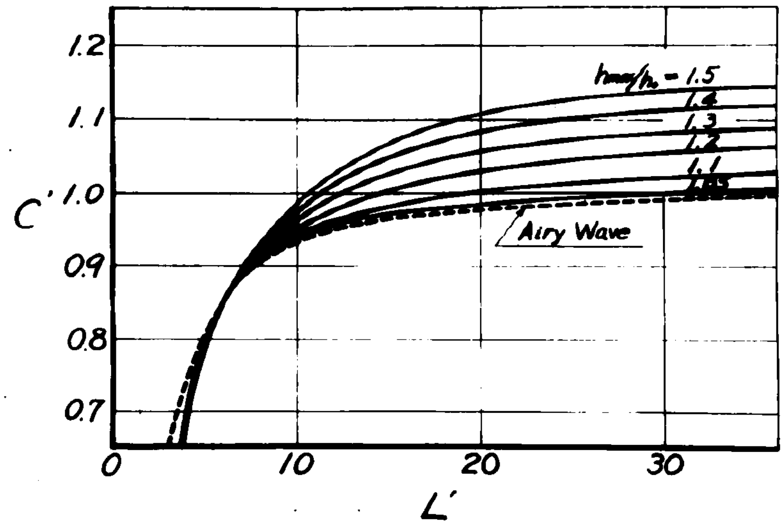

Fig. 1-20 Relationship between dimensionless wave velocity and dimensionless wave length

indicated in the next. Experimental investigation of Keulegan[1)] verifies the analytical conclusion that the value of 10 for L/h_o could be considered the dividing line between the Stokian and the cnoidal waves and for larger values of the ratio cnoidal wave could be applied, and the same conclusion will be illustrated in the figure.

(4) If k becmmes zero, h_{max} and h_{min} also approach the undisturbed level . Eq.(15) is then

$$L = 4(V_w - u_o)h_o/\sqrt{3gh_{max}\cos\theta - 3(V_w - u_o)^2h_o^2/h_{max}h_{min}}\int_0^{\pi/2}\{1 + (k^2/2)\sin^2\chi + \cdots\}d\ ,$$

$$=\{4(V_w - u_o)h_o/\sqrt{3gh_o\cos\theta - 3(V_w - u_o)^2}\}\cdot(\pi/2).$$

For larger waves of $L/h_o > 3.626$, or conversely, shallow water waves, $h_o/L < (1/3.626)$, the above equation becomes

$$V_w = u_o + \sqrt{gh_o\cos\theta\{1 - (4\pi^2h_o^2/3L^2) + (4\pi^2h_o^2/3L^2)^2 + \cdots\}}.$$

and therefore for small waves of small relative depth, the equation is replaced by the following expression with a sufficient accuracy,

$$V_w \doteqdot u_o + \sqrt{(gL\cos\theta/2\pi)\tanh(2\pi h_o/L)}\ . \qquad (17)$$

This equation indicates the velocity of propagation for Airy waves in shallow water and usually applied to the wave for $1/10 < h_o/L < 1/2$.

(c) Hydraulic Characteristics of Solitary Waves

As a special case that the surface undulation becomes still at infinity, the solitary wave is derived from the basic relationship of Eq.(6). At infinity, evidently, $h = h_0$ and $C_1 = (2K^2/h_0) + gh_0^2 \cos\theta$, so that the surface profile equation becomes

$$(dh/dx)^2 = -(3g\cos\theta/K^2)\{h^3 - (K^2/gh_0^2\cos\theta + 2h_0)h^2 + (2K^2/gh_0 \cos\theta + h_0^2)h - K^2/g\cos\theta\} = (3g\cos\theta/K^2)(h - h_0)^2(K^2/gh_0^2\cos\theta - h). \tag{18}$$

The surface profiles vary from the normal depth to the maximum depth, $K^2/gh_0^2\cos\theta$. Denoting the maximum depth by h_{max}, selecting the origin at the apex, Eq.(18) becomes

$$h = h_0 + (h_{max} - h_0)\operatorname{sech}^2[\{\sqrt{3g\cos\theta(h_{max} - h_0)}/2K\}(x - V_w t)], \tag{19}$$

and it represents the wave pattern of solitary wave as the permanent type. Denoting the wave height by a_0, Eq.(19) is transformed into

$$h = h_0 + a_0 \operatorname{sech}^2\sqrt{3a_0/4h_0^2(h_0 + a_0)}\cdot(x - V_w t), \tag{20}$$

and known as the Rayleigh type of solitary wave. If the wave height is small compared with the original depth, Eq.(20) is approximately

$$h = h_0 + a_0 \operatorname{sech}^2\sqrt{3a_0/4h_0^3}(x - V_w t), \tag{21}$$

and known as the Boussinesq-Keulegan-Patterson type. Experimental data of J.W. Daily and S.C. Stephan[16] indicate Eq.(21) can express more precisely the surface patterns of wave than Eq.(20).

When k approaches unity in the expression for cnoidal waves, Eq.(12) becomes

$$h = h_0 + (h_{max} - h_0)\operatorname{sech}^2[\{\sqrt{3g\cos\theta(h_{max}-h_0)}/2K\}(x - V_w t)],$$

and it is equivalent to the expression of solitary wave. Thus, the solitary wave is a special case of cnoidal wave as has been already described by H. Lamb[17], Keulegan and Patterson[11]. The hydraulic characteristics of solitary wave are conclusively indicated in the following[18),19)].

(1) The flow regime, when observed from the moving coordinates, is rapid and this fact is expressed by K.O. Friedlich and D.H. Heyers[20)] for irrotational solitary waves.

(2) The absolute velocity is evidently calculated by the progressive discharge rate and

$$V_w = u_o + \sqrt{gh_{max}\cos\theta}\ , \tag{22}$$

which is identical to the empirical formula of S.Russell and the theoretical result of Boussinesq and Rayleigh and verified as the most suitable formula.

(3) Two components of velocity in still water are expressible as

$$u = c\eta/(h_o + \eta),$$

and

$$v = \sqrt{3}c(a_o/h_o)^{1/2}(1 + a_o/h_o)^{1/2}\cdot(\eta/h_o)(1 + \eta/h_o)^{-2}(1 - \eta/h_o)^{1/2}(y/h_o)\ ,$$

in which y is the distance from the channel bottom.

(4) The volume of a solitary wave per unit width is

$$\forall = (4/\sqrt{3})h_o^2(a_o/h_o)^{1/2}(1 + a_o/h_o)^{1/2}. \tag{23}$$

(5) The kinetic energy of solitary wave is

$$E_k = (1/2)\int_{-\infty}^{+\infty}\rho(u^2 + v^2)(h_o + \eta)dx$$
$$= (2/\sqrt{3})\rho g h_o^3\{2(a_o/h_o)^{1/2}(1 + a_o/h_o)^{1/2}(1 + 2a_o/h_o) - (1 + a_o/h_o)\log|1 + \sqrt{(a_o/h_o)/(1 + a_o/h_o)}|/|1 - \sqrt{(a_o/h_o)/(1 + a_o/h_o)}|\}, \tag{24}$$

and the potential energy is

$$E_p = (1/2)\rho g\int_{-\infty}^{+\infty}\eta^2 dx = (4/3\sqrt{3})\rho g h_o^3(a_o/h_o)^{3/2}(1 + a_o/h_o)^{1/2}. \tag{25}$$

The total energy of solitary wave per unit width is expressed by the sum of Eqs.(24) and (25), that is,

$$E = (4/\sqrt{3})\rho g h_o^3 \{(a_o/h_o)^{1/2}(1 + a_o/h_o)^{3/2} - (1/2)(1 + a_o/h_o) \log|1 + \sqrt{(a_o/h_o)/(1 + a_o/h_o)}|/|1 - \sqrt{(a_o/h_o)/(1 + a_o/h_o)}|\}. \quad (26)$$

When the amplitude is small, $h_o \gg a_o$, Eq.(26) is approximately

$$E = (8/3\sqrt{3})\rho g h_o^{3/2} a_o^{1/2}, \quad (27)$$

and equivalent to the Keulegan formula. It is evident that, in this case of small waves, the potential energy is equal to the kinetic energy as commonly seen in the theory of Airy wave. If the kinetic energy of vertical motion is denoted by E_{kv}, the following relationship between the kinetic and potential energies is obtained.

$$E_k - E_p = 2E_{kv}. \quad (28)$$

This relationship indicates the kinetic energy of finite amplitude waves is larger than the potential energy by twice amount of the kinetic energy of vertical motion, as V.P. Starr[21)] first verified in his gravitational wave theory.

1 - 3 - 4 Attenuation of Solitary Waves

In the foregoing section, it described that the basic flow pattern of transient flow in open channels was expressible as solitary waves in a second approximation. As the solitary wave is of stable form in its surface profile, so it can travel for a long distance without change in its wave form. When the sudden increase of discharge is released in the channel, the translation wave will be travelled downstream in the canal with a form of solitary wave. The influence of released water will reach far downstream until the wave is diminished. The gradual damping process of solitary waves was first treated by G.H. Keulegan[22)] in 1948, who used the small amplitude theory of solitary waves of Boussinesq characterized by equal distributions of kinetic and potential energies and the boundary layer theory expressed in the literature of Lamb[17]. He indicated that the damping coefficient of wave defined as the ratio of wave amplitude to travelled distance was $(1/12)g^{1/2}h_o{}^{3/2}\nu^{-1/2}$. In 1953, Daily and Stephan[16] published an empirical formula for the attenuation, as results of much experimental data. More recently in 1956 and 1957, A.T. Ippen and G. Kulin[23),24] presented results of further experiments and proposed a new approximate attenuation formula based on the Blasius' formula of frictional coefficient experimentally verified by their experiment in the wave flume, and in the same year the author[19] also studied theoretically the attenuation process of solitary waves by means of the theory of laminar layer growth.

In this section, the brief description of attenuation problem of solitary waves in uniform smooth channels will be treated, to give an engineering information for hydraulic design of conveyance structures and canals which will be capable of carriage of sudden increased water.

The rate of energy dissipation when the solitary wave passes the channel is given by differentiating Eq.(26) in 1-3-3 with respect to the time, that is,

$$dE/dt = -(2/\sqrt{3})\rho g^{3/2}h_o^{5/2}\{4(a_o/h_o)^{1/2}(1 + a_o/h_o) - (1 + a_o/h_o)^{1/2} \log|1 + \sqrt{(a_o/h_o)/(1 + a_o/h_o)}|/|1 - \sqrt{(a_o/h_o)/(1 + a_o/h_o)}|\}\cdot d(a_o/h_o)/d(s/h_o) \ , \qquad (29)$$

in which a_o is a wave amplitude at a point under consideration and s is the travel distance of a wave.

The experimental study of Ippen, Kulin and Raza[23] indicates the laminar layer develops near the smooth boundary when passing the wave and the velocity profile in the layer is expressed in terms of parabolic or linear equation when observed from the moving coordinates with the constant celerity. When the parabolic flow is assumed in the layer, the first approximation of energy dissipation rate is

$$dE/dt = 2\mu\int_{-\infty}^{+\infty}(u_0^2/\delta)dx, \tag{30}$$

in which δ and u_0 are thickness of laminar layer and the main flow velocity.

The thickness of laminar layer will be considered rather thin compared with the main potential flow, so that the assumption that the flow is unconfined will be available as a first approximation. The boundary layer equation to evaluate the thickness is then,

$$(u^*/u_0)^2 = (\partial\theta/\partial x) + (1/u_0)(\partial u_0/\partial x)(2\theta + \delta_*) + (1/u_0^2)(\partial/\partial t)(u_0\delta_*), \tag{31}$$

in which θ and δ_* are thicknesses of momentum and displacement. Transforming the coordinates into the moving one, Eq.(31) becomes

$$(u^*/u_0)^2 = (d\theta/dx) + (1/u_0)(du_0/dx)(2\theta + \delta_* - c\,\delta_*/u_0) - (c/u_0)(d\delta_*/dx). \tag{32}$$

Inserting values of θ and δ_* for parabolic flows, Eq.(32) becomes

$$(d\delta^2/dx) + (2/u_0)(5c - 9u_0)(5c - 2u_0)^{-1}(du_0/dx)\delta^2 = -30\nu/(5c - 2u_0). \tag{33}$$

Transforming the independent variable from x to η and introducing the dimensionless parameters $\eta = a_0\sigma$, $\delta = a_0 i$, and $a_0 = h_0\lambda_0$, Eq.(33) becomes

$$(d i^2/d\sigma) + (2/\sigma)(5 - 4\lambda_0\sigma)(1 + \lambda_0\sigma)^{-1}(5 + 3\lambda_0\sigma)^{-1}i^2 = \pm(60/\sqrt{3}R_c)\lambda_0^{-5/2}(1 + \lambda_0\sigma)/\sigma\sqrt{1 - \sigma}(5 + 3\lambda_0\sigma), \tag{34}$$

in which $R_c = g^{1/2}h_0^{3/2}/\nu$.

For small values of λ_0, which indicates small waves, Eq.(34) is approximately

$$(d i^2/d\sigma) + (2/\sigma)i^2 = \pm(12/\sqrt{3}R_c)\lambda_0^{-5/2}/\sigma\sqrt{1 - \sigma}. \tag{35}$$

The solution of positive branch is then under the condition of i_1 = 0 at $\sigma = 0$,

$$\sigma^2 i_1^2 = (16/\sqrt{3}\, R_c)\lambda_o^{-5/2}\{1 - (1 + \sigma/2)\sqrt{1 - \sigma}\}, \tag{36}$$

and that of negative branch is obtained under the condition that $i_1 = i_2$ at $\sigma = 1$,

$$\sigma^2 i_2^2 = (16/\sqrt{3}\, R_c)\lambda_o^{-5/2}\{1 + (1 + \sigma/2)\sqrt{1 - \sigma}\}. \tag{37}$$

The above equation implies that at negative infinity, the layer thickness becomes infinitely large though the actual flow in confined. Therefore, Eq.(37) is applied to the point where the thickness of layer is equal to the water depth, and the point is approximately calculated by

$$\sigma_c = (4\sqrt{2}\, /\sqrt[4]{3}\,)R_c^{-1/3}\lambda_o^{-1/4}. \tag{38}$$

The total rate of energy dissipation within the laminar layer is then

$$\begin{aligned}(dE/dt) &= 2\mu\int_{-\infty}^{(x)_{\sigma=\sigma_c}} u_o^2(y_o + \eta)^{-1}dx + 2\mu\int_{(x)_{\sigma=\sigma_c}}^{0} u_o^2\delta_2^{-1}dx + 2\mu\int_0^{\infty} u_o^2\delta_1^{-1}dx,\\ &= 2\mu\left(\int_{-\infty}^{0} u_o^2\delta_2^{-1}dx + \int_0^{+\infty} u_o^2\delta_1^{-1}dx\right) - 2\mu\int_{-\infty}^{(x)_{\sigma=\sigma_c}} u_o^2\{(1/\delta_2) - 1/(y_o + \eta)\}dx.\end{aligned} \tag{39}$$

Transforming the independent variable from x to η, and introducing the dimensionless parameters as described in the foregoing, the [illegible] energy dissipation rate is

$$(dE/dt) = (1/3^{1/4})\mu g h_o R_c^{1/2}\lambda_o^{7/4}(1 + \lambda_o)^{3/2}\Big[\int_0^1 \sigma\sqrt{2 + \sigma\sqrt{3 + \sigma}}\,/\{\sqrt{3 + \sigma}\sqrt{1 - \sigma}(1 + \lambda_o\sigma)^2\}d\sigma - \int_0^{\sigma_c} \sigma/\{\sqrt{1 - \sigma}(1 + \lambda_o\sigma)^2\}\cdot d\sigma\{(\sigma/2)\sqrt{1 + (1 + \sigma/2)^{-1/2}(1 - \sigma)^{-1/2}}\} - (2/3^{1/4})R_c^{-1/2}\lambda_o^{-1/4}(1 + \lambda_o\sigma)^{-1/2}\Big]. \tag{40}$$

as $u_o = c\eta/(h_o + \eta)$, $c = \sqrt{g(h_o + a_o)}$, and

$$(d\eta/dx) = (1/h_o)\sqrt{3a_o/(h_o + a_o)}\,\eta\sqrt{1 - \eta/a_o}.$$

The energy dissipation of a solitary wave when passing through the channel is expressed in Eq.(29), which is transformed into

$$(dE/dt) = -(4/3^{1/2})\rho g^{3/2} h_o^{5/2} \lambda_o^{1/2} (1 + 1.6667\lambda_o + 0.1333\lambda_o^2 + \cdots) d\lambda_o / d(s/h_o). \tag{41}$$

Combining Eqs.(40) and (41), the attenuation of wave height is obtained as a function of the distance travelled, and the approximate expression is then

$$\lambda_o^{-5/4}(1 + \lambda_o)^{-3/2}(1 + 1.6667\lambda_o + 0.1333\lambda_o^2 + \cdots)(1 - 2.2520\lambda_o - 6.4884\lambda_o^2 - \cdots)^{-1} d\lambda_o + (3^{1/4} \cdot 1.2889/4) R_c^{-1/2} d(s/h_o) = 0, \tag{42}$$

and finally, the relationship for the attenuation of solitary wave is

$$\lambda_o^{-1/4}(1 - 0.8062\lambda_o - 1.6348\lambda_o^2 - 5.0973\lambda_o^3 - \cdots) - \lambda_i^{-1/4}(1 - 0.8062\lambda_i - 1.6348\lambda_i^2 - 5.0973\lambda_i^3 - \cdots) = 0.1060 R_c^{-1/2}(s/h_o), \tag{43}$$

in which λ_i is the initial value of dimensionless wave height.

This is the two dimensional attenuation formula based on the distance traversed. If the channel is of limited width, however, some factors to include the frictional effects along the side wall must be added. If $(1 + 2h_o/b)$ is used as a first approximation, which is only valid for very low waves, the following relationship for rectangular channel is obtained.

$$\lambda_o^{-1/4}(1 - 0.8062\lambda_o - 1.6348\lambda_o^2 - 5.0973\lambda_o^3 - \cdots) - \lambda_i^{-1/4}(1 - 0.8062\lambda_i - 1.6348\lambda_i^2 - 5.0973\lambda_i^3 - \cdots) = 0.106 R_c^{-1/2}(1 + 2h_o/b)(s/h_o). \tag{44}$$

In the same way, for the case of Couette flow, the attenuation equation is obtained. In this case, the coefficient of damping is 0.0918.

The damping coefficient K' is defined by Keulegan as follows.

$$K' = (\lambda_o^{-1/4} - \lambda_i^{-1/4}) / R_c^{-1/2}(1 + 2h_o/b)(s/h_o). \tag{45}$$

Fig. 1-21 indicates the behaviours of damping coefficient with the

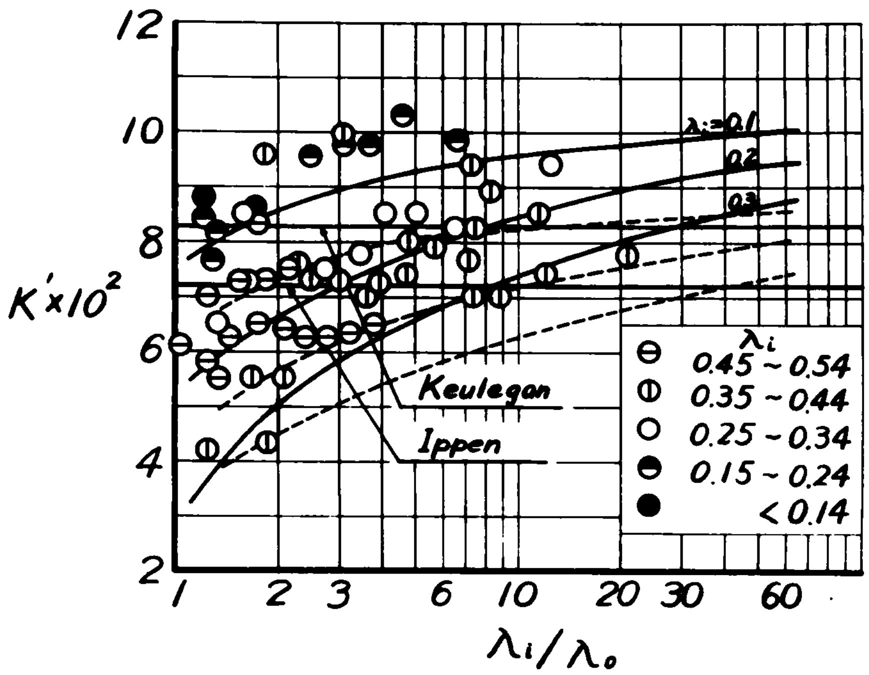

Fig. 1-21 Damping coefficient of solitary waves in smooth rectangular channels **(solid: parabolic, dotted: Couette)**

parametric notation λ_i. In the same figure, experimental data of Russell[22)] and Ippen[23)] are also plotted. In Russell's data, the temperature of water is assumed to be 15.8°C by Keulegan and in Ippen's data the kinematic viscosity is assumed to be 10^{-5} ft^2/sec. The agreement between theoretical curve and experimental data are not close, but a similar tendency in the behaviour will be seen. It is rather surprising that the approximate calculation described in the foregoing will indicate a similar behaviours in wave attenuation. It is constant for the small amplitude theory, that is 1/12 for Keulegan's expression and 1/14 for Ippen's expression.

Although this approach to attenuation implies many physical assumptions which will influence the development process of laminar layer to a considerable extent, it can not be ascertained whether these assumptions are valid or not, because of insufficient accuracy of measuring basic data. After establishment of further progress in experimental measurement, a discussion will be again hopeful.

---

1) Keulegan, G.H., Wave Motion, Engineering Hydraulics, edited by H. Rouse, John Wiley, New York, 1950.

2) Iwagaki, Y., and Iwasa, Y., On the Hydraulic Characteristics of the Roll-Wave Trains, Studies on the Thin Sheet Flow, 7 th Report, Jour. JSCE, Vol. 40, No. 1, Jan. 1955 (in Japanese).

3) Frank, J., Nichtstationare Vorgange in den Zuleitungs und Ableitungskanalen von Wasserkraftwerken, Springer, Berlin, 1957.

4) Rich, G.R., Hydraulic Transients, McGraw-Hill, New York, 1951.

5) Favre, H., Étude théorique et expérimentale des ondes de translation dans canaux découverts, Dunod, Paris, 1935.

6) Massé, P., Hydrodynamique fluviale, 1935.

7) Massau, J., Mémoire sur l'integration graphique des equations aux derivees partielles, Annales de l'Association des Ingenieurs sortis des Écoles Speciales de Gand, Vol. 23, 1900.

8) Ré, R., Étude du lacher instantane d'une retenue d'eau dans un canal par la methode graphique, La Houille Blanche, Vol. 1, 1946.

9) Stoker, J.J., Water Waves, Interscience, New York, 1957.

10) Doeringsfeld, H.A., and Barker, C.L., Pressure-Momentum Theory Applied to the Broad-Crested Weir, Proc.ASCE, Vol. 65, 1939.

11) Keulegan, G.H., and Patterson, G.W., Mathematical Theory of Irrotational Translation Waves, Jour. NBS, RP 1273, Vol. 24, 1940.

12) Benjamin, T.B., and Lighthill, M.J., On Cnoidal Waves and Bores, Proc. Roy. Soc. A, Vol. 224, 1954.

13) Packam, B.A., The Theory of Symmetrical Gravity Waves of Finite Amplitude, II. The Solitary Wave, Proc. Roy. Soc. A, Vol. 222, 1952.

14) Sérre, F., Contribution à l'étude des écoulements permanents et variables dans les canaux, La Houille Blanche, June-July, 1953.

15) Iwasa, Y., Analytical Consideration on Cnoidal and Solitary Waves, Memoirs, Fac. Eng., Kyoto University, Vol. 17, No. 4, Oct. 1955.

16) Daily, J.W., and Stephan, S.C., The Solitary Waves, Proc. 3 rd Conference Coast. Eng., 1953.

17) Lamb, H., Hydrodynamics, Dover, New York, 1932.

18) Iwasa, Y., Hydraulic Characteristics of Solitary Waves, Abstract, Proc. Annual Conv. JSCE, May 1956 (in Japanese).

19) Iwasa, Y., Attenuation of Solitary Waves on a Smooth Bed, Jour. Hydraulics Division, Proc. ASCE, HY 3, June 1957.

20) Friedlich, K.O., and Heyers, D.H., The Existence of Solitary Waves, Comm. Pure and Appl. Math., Vol. 7, No. 3, Aug. 1954.

21) Starr, V.P., Momentum and Energy Integrals for Gravity Waves of Finite Height, Jour. Marine Res., Vol. 6, No. 3, 1947.

22) Keulegan, G.H., Gradual Damping of Solitary Waves, Jour. Res. NBS, RP 1895, Vol. 40, 1948.

23) Ippen, A.T., Kulin, G., and Raza, M.A., Damping Characteristics of the Solitary Wave, MIT Hydrodynamics Lab. Technical Report No. 16, Apr. 1955.

24) Ippen, A.T., and Kulin, G., The Effect of Boundary Resistance on Solitary Waves, La Houille Blanche, No. 3, Jul.-Aug. 1957.

II. TRANSITIONAL BEHAVIOURS OF FLOW CHARACTERISTICS OF GRADUALLY VARIED FLOWS IN CHANNEL TRANSITIONS AND CONTROLS

1. General Scope of Problems

The engineering development of hydraulic projects for various forms of conveyance of water requires careful analysis of hydraulic problems. Detailed problems on particular projects may vary with local conditions, owing to their peculiarities. The general principle, however, available to insure the successful establishment is similar on most gravity projects. Following hydraulic considerations of the conveyance of water are needed for these major purposes, as described by I.E. Houk[1).

(1) To work out alternative designs for divergent types and locations of conduits of equivalent capacities, for use in determining the most desirable efficient and economical construction.

(2) To insure that the conduits as finally located and designed will safely and efficiently carry the maximum rates of flow that will be needed.

(3) To insure that conditions of flow through the conduits, during both partial and full stages of project development, will be such as to reduce future cost of operation and maintenance as much as possible.

Hydraulic analysis for engineering purposes may involve various fields of analytical and experimental methods of one or more characteristics of flow. For examples, the prediction of the water surface profile for a particular design discharge typically classified as the hydraulics of gradually varied flow, the analysis of hydraulic jump, the establishment of functional varieties of control structures like weir and spillway in the hydraulics of rapidly varied flow, and the transient problems in open channel are included.

Restricting the problem only to the steady flow characteristics in the present part, it is commonly subdivided into two categories of gradually and rapidly varied flows. The gradually varied flow is defined as the flow in which the changes are anything but sudden, following by the description of C.J. Posey[2]. The main purpose of the part is therefore to study the behaviours of steady flows, in which gravity and boundary resistance will be of major importance, and the prediction of flow pattern for particular discharges and channel geometries.

The mathematical analysis of back water curves, subjected by very large radii of curvature, forms the basis for the classification of all the possible types of water surface profiles in gradually varied flows. Since works of Dupuit and Bresse, many attempts have been made either to clarify the behaviour of surface profiles for various channel shape and grade or to establish the tabulation of back water functions under certain assumptions. In 1930 - 40, the Manning formula first was used to calculate the back water function, while earlier methods had used only the formula of Chézy. In 1955, V.T. Chow[3] improved the theory and made fruitful tabulation of back water function for all shapes of channel geometry.

The hydraulic jump, which is a stationary discontinuity , is occured usually at the transition from shooting to tranquil. It is commonly characterized by the rapidly varied flow owing to its sudden change in the water surface profile and that the characteristics are solved by means of the momentum conservation law. However, this type of transition is frequently observed when proceeding the surface profile of gradually varied flow, and without the knowledge on the hydraulic jump, the complete prediction of surface profiles of water can not be obtained, so that it must be considered together with the gradual variation in flow profiles. When the upstream depth of water becomes **critical**, the hydraulic jump still is seen, but its form indicates to be undular with a train of successive waves and the phenomenon is known as the undular jump. The common hydraulic jump is calculated by the momentum and mass conservation laws, while the analysis of undular jump phenomenon is difficult to treat, even with the introduction of the vertical acceleration of the flow.

The brief description in the foregoing concerned with two cases of flow characterized by flow regimes. The first is that no singular point in the basic equation of surface profile will be observed in particular reaches. The surface profile is proceeded by the basic features of open channel flows, which indicates the tranquil flow can bring a small disturbance upstream while the rapid flow can not to the upstream direction. The second case represented by the hydraulic jump is that the denominator in the basic equation becomes zero.

On the other hand, when the channel geometry, channel roughness, and grade are changed as in natural channels and even in artificial watercourses, the condition that both numerator and denominator in Eq.(1) in 1-2-2 become simultaneously zero, or the normal flow curve and the critical depth curve intersect together at a point, may be occured for a particular discharge. This transitional characteristics have generally been discussed in the previous part and the results obtained are expressible as

(1) Transition from tranquil to rapid is occured at a saddle point;

(2) Transition from rapid to tranquil, genrally through the hydraulic jump, is produced by nodal and focal points.

Furthermore, at the transitional point, which is commonly a saddle point, the most significant relationship between the discharge and the energy known as the Bélanger-Böss theorems or the generalized theory of Jaeger is mathematically verified.

In this part, first is dealed the historical development of the hydraulic study in gradually varied flows, which were mostly advanced by many scientists and engineers in the 19 th century. The next chapter concerns with the hydraulic characteristics of jump in a conclusive form. The hydraulic jump is usually classified as a typical example of rapidly varied flow in its flow behaviours. Nevertheless, the basic concept of the jump is needed for the calculation of surface profiles of gradually varied flows. The transitional behaviours of gradually varied flows which are classified by variations of channel geometry, its grade and channel resistance are followed. This chapter is the main subject of the whole part, as the surface profiles are essentially furnished by the theory of transitional characteristics. The last section of the chapter treats with the application of the transitional characteristics to the design problem in hydraulic engineering. The common calculation procedure of water surface curves is the step by step method in uniform and especially non-uniform channels. If the theory of transitional characteristics is not concerned, much errors associated with the procedure are resulted. In gradually varied flows, it becomes important for the estimation of back water zone in reservoirs and the discharge measurement, which is treated in the later part.

---

1) Houk, I.E., Irrigation Engineering, Vol. 2, Projects, Conduits, and Structures, John Wiley, New York, 1956.

2) Posey, C.J., Gradually Varied Channel Flows, Engineering Hydraulics, edited by H. Rouse, John Wiley, New York, 1950.

3) Chow, V.T., Integrating the Equation of Gradually Varied Flow, Proc. ASCE, Separate 838, 1955.

2. Historical Background of Hydraulics of Gradually Varied Flows

2 - 2 - 1 Historical Development of Studies of Gradually Varied Flows in Uniform Channels

The oldest research project as the science of fluid flows in open channels since the middle part of 19 th century was the hydraulics of gradually varied flows in uniform channels. Before the birth of theoretical development in classical hydraulics, the basic knowledge of open channel flows and the resulting design criteria were mainly due to the concept of uniform flow. Uniform flow, however, rarely occures in natural channels because of continuous changes in depth, width and slope along the channel. While the simplest design for man-made channels like irrigation canals and other artificial watercourses provide a uniform cross-section and constant slope, it is not in common feasible owing to topographical and geological conditions. Hydraulic engineers, therefore, must concern with non-uniform flows. The types of water surface profiles for steady flow are often called as backwater curves.

The methods of analysis for gradually varied flows are the following two procedures: One is the step by step method of integration of original surface profile equation based on the Bernoulli theorem of energy or the momentum conservation law. It is a common method of calculation applicable to all shapes in channel section and especially in non-uniform channels. The other is analytical one furnished with the exact integration of original equation. As the surface profile equation can not be explicitely expressed in terms of the depth for all types of channel shape, so it is considered as a particular case of the former procedure. For the purpose of complete description of surface profiles, the earlier studies have been mostly developed for channels of a specific cross section like two dimensional, rectangular and so on. In 1932, B.A. Bakhmeteff[1] first extended the classical theory to the refined form in all shapes of channels and thereafter many hydraulic engineers have been enforced to obtain a renewed and complete form of characteristics of varied flow function. The historical development of the present study since the work of Dupuit in 1848 is seen in the literature of Chow[2]. Recently, Chow[2] proposed a refined integrating method with a tremendous amount of labours and obtained a table and a chart which provide the hydraulic engineer a time-saving and simple calculation procedure for practical engineering problems.

In the foregoing discussion, the hydrostatic law in pressure is substantially assumed and the surface curvature of flow is also ignored. The accuracy of the elementary theory developed by many investigators is definitely improved by the inclusion of more complete assumption of basic flow behaviours regarding pressure and surface curvature. As **briefly** described in the former part, Boussinesq[3] first studied the higher order behaviours of open channel flows characterized by the non-hydrostatic pressure. Inserting Eq.(36) into Eq.(73) of the former part, and expressing the shear function in terms of the Chézy law, the basic equation of Boussinesq becomes

$$(d^3h/dx^3) + 3(gh/q^2 - 1/h^2)(dh/dx) + 3g(1/C^2h^2 - ih/q^2) = 0. \quad (1)$$

In the above non-linear equation of third order, the correction factor of momentum β is added to the original equation, following the expression of Boussinesq. If the velocity distribution is considered non-uniform, as seen in the notation of β , the pressure distribution is also influenced by the velocity profile, and therefore the constant coefficient of 3 must be changed.

Owing to great mathematical complexity resulted from the non-linearity, Boussinesq analyzed the hydraulic characteristics of the linearized version and divided into two categories, depending on the channel slope i,

$$i \gtrless (g/C^2)\{1 \mp 3(g/C^2)^{2/3}\}. \qquad (2)$$

In the first case of mild channels, the transition from non-uniform to uniform flow is produced and the formation of a series of successive waves is also resulted, whereas the opposite transition takes place without waves. On the contrary, in the second case of steep channels, uniform flow reaches gradually from non-uniform flow upstream.

Following the monumental work of Boussinesq, Fawer[4)] also studied the same subject as Boussinesq did, with the use of energy equation, and classified the surface profiles of gradually varied flows of appreciable curvature, with result of further anticipation of flow behaviours in terms of the elliptic function of Jacobi.

Recently, F.Sérre[5)] investigated systematically the higher order behaviours of open channel flows by means of both approaches of energy and momentum. His mathematical deduction is very skillful, but not referred to the theory of ordinary differential equation.

With respect to this problem, the basic equations of gradually varied flows by the momentum approach are

$$(1/h)(dM_0/dx) = \sin\theta - q^2/C^2h^3,$$

and

$$M_0 = (q^2/gh) + (h^2\cos\theta/2) + (q^2/3g)(d^2h/dx^2) - (q^2/3gh)(dh/dx)^2,$$

from Eqs.(36) and (73) in Part I. Introducing $p = dh/dx$ in above equations, the following equations are obtained.

$$(dM_0/dx) = h(\sin\theta - q^2/C^2h^3),$$

$$(dh/dx) = p,$$

$$(dp/dx) = (3gM_o/q^2) - (3/h) - (3gh^2\cos\theta/2q^2) + (p^2/h).$$

The location of singular point, which is determined by $dM_o/dx = 0$, $dh/dx = 0$ and $dp/dx = 0$, thus is for Chezy flows,

$$h_c^3 = q^2/C^2\sin\theta \ , \quad p_c = 0, \text{ and } M_{oc} = (q^2/gh_c) + (h_c^2\cos\theta/2).$$

This point indicates apparently the uniform depth. Transforming the origin to the singular point, and putting

$$h = h_c + \eta, \quad p = p_c + \rho, \text{ and } M_o = M_{oc} + m,$$

the linearized versions of the system of equations are

$$(dm/dx) = 3i\cos\theta\cdot\eta,$$

$$(d\ \ /dx) = p,$$

and

$$(dp/dx) = (3i_{cr}/ih_o^3\cos\theta)m + (3/h_o^2)(1 - i_{cr}/i)\eta,$$

in which i_{cr} is the critical slope of g/C^2 and i the channel slope of $\tan\theta$. Again introducing the dimensionless parameters of $x = h_c\xi$ and $\eta = h_c\zeta$, and combining above equations, the following linearized equation of third order for the dimensionless depth is obtained.

$$(d^3\zeta/d\xi^3) + 3(i_{cr} - i)/i\cdot(d\zeta/d\xi) - 9i_{cr}\zeta = 0. \qquad (3)$$

If three roots of Eq.(3) are assumed $_1$, $_2$, and $_3$,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0, \text{ and } \lambda_1\lambda_2\lambda_3 = 9i_{cr},$$

so that one real and positive root, say λ_1, must be existed. Therefore, the following two relations will be deduced, depending on the mathematical behaviour of discriminant.

(1) $\lambda_1 > 0$, $\lambda_2, \lambda_3 < 0$,

(2) $\lambda_1 > 0$, $\lambda_2 = u + iv$, $\lambda_3 = u - iv$, $u < 0$,

where u and v are real. In the first case, the singular point is classified as nodal saddle point and in the second case focal saddle point. The discrimination between two cases is in the following.

$$(81/4)i_{cr}^2 + (i_{cr} - i)^3/i^3 \gtrless 0.$$

If the adverse slope is not considered, the above inequality yield the following relation for discrimination, as $3^3\sqrt{3/4}\cdot i_{cr}^{2/3} > 0$.

(1) $i < i_{cr}/\{1 - 3(3/4)^{1/3} i_{cr}^{2/3}\}$.

The singular point, which also indicates the normal flow condition, is classified as focal saddle point. As seen in Fig. 2-1, which indicates the discrimination between two cases, the focal saddle point is commonly occured in the mild channel, and the zone of focal saddle points stretches to the steep channel with the increase of boundary resistance.

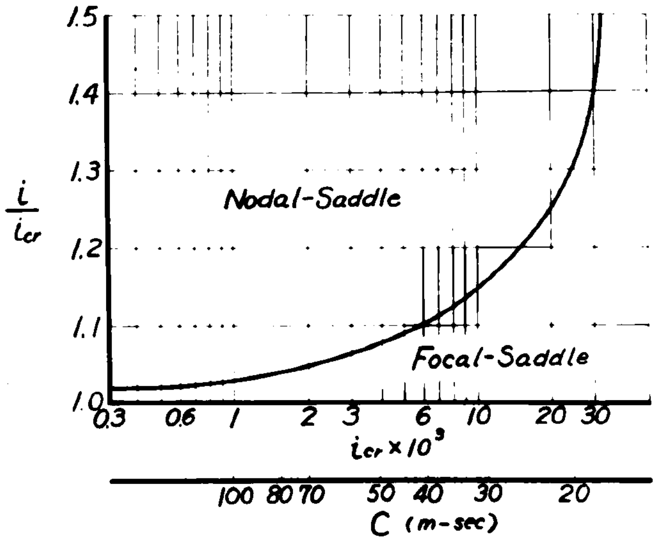

Fig. 2-1 Relationship between i and i_{cr} to discriminate two classes of singular points

The transitional characteristics of flows by this type of singular point is expressed as follows.

When the flow stage is regulated by a control structure like dams and free overfalls, the surface profile gradually approaches uniform in the direction of upstream without wave formation, whereas the downstream surface profile from a gate approaches uniform with a series of waves. The wave pattern described in this section will be sinusoidal owing to the linearization of basic equations. The actual wave profile will be cnoidal in terms of the cn-function of Jacobi as will be indicated in the next section. The basic characteristics described are identical to the analysis of Boussinesq.

(2) $i > i_{cr}/\{1 - 3(3/4)^{1/3} i_{cr}^{2/3}\}$,

The singular point is known as nodal saddle, and apparently Eq. (3) has one real positive root and two real negative roots, for this range of steep slopes. The transition from non-uniform to uniform flow takes place without formation of waves.

The foregoing analysis indicates that the non-uniform tranquil flow is characterized by a series of waves in its surface configuration while the shooting flow without waves. It is often observed the wave formation in open channel flows is appreciable under the condition of near critical regime. Some physical explanation for this fact will be briefly described.

When the surface profiles are of wave form, Eq.(3) has three roots of one positive and conjugate complex. From the relationship between the root and the coefficient of Eq.(3),

$$u = -(\lambda_1/2), \text{ and } v = \sqrt{(9i_{cr}/\lambda_1) - (\lambda_1^2/4)},$$

in which λ_1 is a positive root. The dimensionless wave length, L/h_c, as the solution of Eq.(3) is therefore

$$(L/h_c) = 2\pi/\sqrt{(9i_{cr}/\lambda_1) - (\lambda_1^2/4)}, \tag{4}$$

and Eq.(4) is valid for $i \to i_{cr}/\{1 - 3(3/4)^{1/3} i_{cr}^{2/3}\}$.

The behaviour of (L/h_c) for $i \to i_{cr}/\{1 - 3(3/4)^{1/3} i_{cr}^{2/3}\}$ will be first treated. As $(u^2 + v^2) = (9i_{cr}/\lambda_1)$, so the following relationship for λ_1 is obtained.

$$\lambda_1^3 - 3(1 - i_{cr}/i)\lambda_1 - 9i_{cr} = 0. \tag{5}$$

in the original equation, $i = i_{cr}/\{1 - 3(3/4)^{1/3} i_{cr}^{2/3}\}$ makes roots real and equal. From the discriminant of Eq.(5), the condition for equal roots is evidently

$$(81/4)i_{cr}^2 + (i_{cr} - i)^3/i^3 = 0,$$

and it is equivalent to the condition of discrimination for the original equation, and consequently the denominator of Eq.(4) becomes zero. Thus, the wave length of surface profiles approaches quite large to infinity as $i \to i_{cr}/\{1 - 3(3/4)^{1/3} i_{cr}^{2/3}\}$.

If (i/i_{cr}) is very small, the real root is approximately expressed as

$$\lambda_1 \doteqdot 9i_{cr}(i/i_{cr})\{1 - (i/i_{cr})\}^{1/2},$$

so that λ_1 approaches zero, as $(i/i_{cr}) \to 0$, and consequently the wave length expressed by Eq.(4) also becomes zero.

Between both limiting cases for (i/i_{cr}), the wave length is finite and it is easily seen the surface profiles of tranquil flow near critical regime is of distinct wave form by eyes, and therefore it will be observed that the wave formation of flow near critical regime is apparent. Furthermore, the very important conclusion for the wave formation will be derived. When the channel characteristics approaches a limiting case for discrimination, the wave length also becomes quite large, so that it is expected that the disturbance in itself is of quite large in wave length. In the preceding part concerned with the hydraulic instability, it is found the condition of flow stability is basically due to its wave length in its essential character. If the flow character described here is conjectured to be the same property as that of shooting flow explained in the preceding part, the Vedernikov criterion for initial instability of rapid flow will be valid for all cases, though no informations on the character have been made. Further hydrodynamic verification is needed for this important conclusion with the measurements of flow character itself.

2 - 2 - 2 Hydraulic Studies of Gradually Varied Flows in Non-Uniform Channels

The preceding section concerned with the flow behaviours of gradually varied flows of first and second approximations developed by many scientists and engineers in open channels of uniformity in channel geometry. As natural channels or artificial watercourses involve frequent changes of channel geometry and grade as well as channel roughness, so the surface profile of steady flows in non-uniform channels can not be computed by the methods briefly described in the preceding section. As have often indicated, the normal depth curve, which is an asymptote of surface profile for particular discharges, is of less importance for the computation of surface profiles, while the critical depth curve becomes significant for the calculation of surface profiles in non-uniform channels as well as in uniform channels, as it determines the regime of flow, tranquil or rapid, and thus the direction for calculation. Evidently, when the tranquil flow, of which depth is greater than critical at all points in a reach, the surface profile is subjected to downstream control, and the shooting flow as a counterpart of tranquil flow in open channels induces the profile to upstream control.

In non-uniform channels, the water surface profiles for a particular discharge are determined by calculating separately and successively the variation in surface elevation in each of a large number of reaches, which are divided from the whole part so short as to reduce errors of calculation resulted from the approximation in the basic relationship to a sufficient magnitude for engineering purpose, and therefore the choice of reaches must be carefully made. If a reach selected is assumed rather uniform, various procedure described in the preceding section will be approximately used.

Although a broad variety of computation procedures for surface profiles have been proposed by many hydraulic engineers, the basic relationship of calculation is definitely based on the Bernoulli or the momentum theorem, as shown in 1-2-2. The usual method of calculation is a numerical analysis procedure widely known as the standard step method, and it provides the surface profiles of gradually varied flows by the finite difference method without negligence of terms in the basic equation. Following by the energy approach of one dimensional method, expressed by Eq.(63) in 1-1-4.

$$(d/dx)\int(u^2/2g + p/\rho g + y\cos\theta + z)u\,dA = -(\tau s/\rho g)u_b.$$

In the gradually varied flow, the pressure is assumed hydrostatic, so that the above equation becomes

$$dZ = -(Q^2/C^2RA^2)dx - (Q^2/2g)d(1/A^2), \qquad (6)$$

in which $Z = h\cos\theta + z$, $(\tau/\rho gR)(u_b/u_m) = (Q^2/C^2RA^2)$ and the velocity distribution is assumed constant throughout the section. In a reach between two sections of n and (n + 1), Eq.(6) is

$$Z_{n+1} - Z_n = -(Q^2/C_m{}^2R_mA_m{}^2)(x_{n+1} - x_n) - (Q^2/2g)(1/A_{n+1}{}^2 - 1/A_n{}^2), \qquad (7)$$

where C_m, R_m and A_m are avarage values in a reach. If the Chézy law is applied to the computation, C is constant, while $C = (R^{1/6}/n)$ for the Manning flow, and the latter formula is commonly used for calculation. In the computation with the use of tabulation, $(Q^2/C_m{}^2R_mA_m{}^2)$ is called as the friction slope. By means of Eq.(7), the surface profile of gradually varied flow is computated, depending on the flow condition, tranquil or shooting, which is the basic principle of open channel flow, and if the control is occured in a reach, it is evidently a starting point for computation. However, by the existence of control sections, the flow behaviour is characterized by the transitional properties of flow and the secondary influence of non-hydrostatic pressure of flows near critical regime. As seen in Eq.(7), the accuracy of computation largely depends on selected reaches, resulting changes in channel geometry, irregularity of grade, roughness and so on. The distance of reach must be selected with respect to local geometrical and channel conditions. With the increasing request for more accuracy in the final result of surface profiles of water, the troublesome computation for a large number of reaches in a whole part is urged to become necessary. In some special cases, however, the computation procedure will be simplified without being less in accuracy.

If the channel geometry is assumed relatively uniform, the simplified calculation method by means of

$$Z_{n+1} - Z_n = -(Q^2/C_m{}^2 R_m A_m{}^2)(x_{n+1} - x_n), \tag{8}$$

will become appropriate, as the change of velocity head becomes of less importance to the computation procedure for surface profiles. The above simplified method is also available for the reservoir routing, except near the transition reach from reservior to original channel as the engineering approximation.

As one of particular simplified method as same as the preceding one, Grimm's step method will be used and especially it is available for the surface profiles of the stream in its natural state, without the back water effect. In this method, the velocity head is ignored, so that the computation of flows in which the large magnitude of velocity head exists, is considered not to be available.

The preceding method of numerical analysis involves often so troublesome labours for the computation to trace the surface profile, so that the graphical integration methods have also been developed by a large number of engineers. The most simplified method with the graphical aid is expressed as follows. The surface profile equation of gradually varied flows is in a form of $(dh/dx) = g(h, x)$. If the flow behaviour of $g(h, x)$ at all points in the whole channel under investigation is obtained against h, the surface profile will be easily traced by multiplying Δx to $g(h, x)$ in the possible direction of surface tracing. Nearly all graphical methods are essentially based on the above description. Famous procedures are the Escoffier-Raytchine-Chatelain method[6] and the method of Silber[7]. The graphical method of the former is provided by making curves of the sum of velocity head and friction loss for all reaches. Eq.(6) is transformed into, for Manning flows,

$$dZ = -(Q^2/2)(n^2/R_{n+1}{}^{4/3}A_{n+1}{}^2 + n^2/R_n{}^{4/3}A_n{}^2)dx - (Q^2/2g)(1/A_{n+1}{}^2 - 1/A_n{}^2).$$

Bearing in mind that the usual back water calculation is proceeded in the upstream direction, as the flow is subjected to downstream control, the above equation is

$$Z_{n+1} - Z_n = -Q^2\{(F_2)_{n+1} - (F_1)_n\}, \qquad (9)$$

in which

$$F_{1,2} = (1/2gA^2) \pm (L/2K^2), \quad \text{and } K = (1/n)R^{2/3}A.$$

Curves of F_1 and F_2 are determined at all points before calculation. In F_1-Z plane, with the use of a curve of gradient, $-Q^2$, drawn from an initial point, the desired water elevation will be estimated by an intersecting point between the above line and the F_2-curve, so that the surface profile will be successively traced.

More recently, R.Silber[7] proposed a refined procedure of graphical integration for surface profiles by means of the universal characteristics of flow. In Eq.(68) of 1-1-4, the assumption of gradually varied flows makes the relationship as

$$H_o = h + (Q^2/2gA^2),$$

when the channel grade is small. Expressing for the discharge,

$$Q = A\sqrt{2g(H_o - h)}\ .$$

Here, the following dimensionless parameter will be introduced.

$$h^* = h/H_o, \quad q^* = (Q/l_m)(1/H_o)(2gH_o)^{-1/2},$$

in which l_m is the mean width of channels. The dimensionless discharge is then

$$q^* = h^*\sqrt{1 - h^*} ,$$

and q^* is explicitely determined by h^*, and defined as the universal characteristics (caracteristique universelle) by him. The change of total head is from Eq.(69) in 1-1-4,

$$(dH_o/dx) = \sin\theta - (Q^2/C^2RA^2)(u_b/u_m) = \sin\theta - (Q^2/C^2RA^2),$$

so that it is also determined as a function of water depth for a particular section. If the curve is shifted by $\sin\theta$, it indicates the change of water elevation from a reference line in the energy approach. Therefore, it, for a particular reference head H_o', curves of universal characteristics as a parameter of energy loss H_o/H_o' are plotted, the surface profiles are readily traced by the graphical procedure. This method is especially available for understanding of the general features of surface profiles for particular discharges in non-uniform channels. The accuracy, however, is limited owing to the graphical representation and particularly near the singular points, which will be frequently appeared in such channels, the desirable solution can not be expected.

---

1) Bakhmeteff, B.A., Hydraulics of Open Channels, McGraw-Hill, New York, 1932.

2) Chow, V.T., Integrating the Equation of Gradually Varied Flow, Proc. ASCE, Separate 838, 1955.

3) Boussinesq, J.V., Essai sur la théorie des eaux courantes, Mémoirs présentés par divers savants à l'Académie des Sciences, Paris, 1877.

4) Fawer, C., Étude de quelques écoulements permanents à filet coubes, Thesis, Lausanne, 1937.

5) Sérre, F., Contribution à l'étude des écoulements permanents et variables dans les canaux, La Houille Blanche, June - July 1953.

6) Stipp, J.S., Backwater Profiles Solved by Escoffier-Raytchine-Chatelain Method, Civil Engineering, Aug. 1953.

7) Silber, F., Étude et trace des écoulements permanents en canaux et rivières, Dunod, Paris, 1954.

3. Hydraulic Characteristics of Hydraulic Jump

2 - 3 - 1 General Remarks

When a flow involves a discontinuity in hydraulic characteristics, the discontinuity can travel up- and downstream, as has been indicated in 1-1-6, and the ascending wave is only considered, the velocity propagation is given by Eqs.(120) and (121) in 1-1-6. If the absolute velocity of ascending wave becomes zero, the discontinuity front can not travel in the upstream direction and will remain in a certain form at the point where $V_W = 0$. This phenomenon is called as a hydraulic jump and often used for a mean of energy dissipation by hydraulic engineers. Limiting the problem in the steady flow, the hydraulic jump also may be resulted in a channel, when a discontinuity in channel geometry exists or the underflow from a gate is effluxed in mild and horizontal channels. This phenomenon is apparently seen in the transition from shooting to tranquil flow as the flow over a spillway apron. The Froude number in the rapid flow is not so large over unity, the hydraulic jump takes a formation of successive waves of damping wave height, whereas for high Froude numbers it becomes an abrupt change in flow behaviour with powerful turbulence and horizontal rollers and with a foaming water surface. Herewith, a question, of which form is more likely, will be arised. This problem is so difficult that the complete solution has not been achieved, but it seems the problem is analogous to the transonic one in mathematical form and the hydrodynamic stability in physical behaviour, and the former jump is known as the normal jump and the latter undular. The normal jump is described in most hydraulic texts and often used as the energy dissipator through the transition from rapid to tranquil flow, in which the velocity is so low that it will either not scour the stream bed or the bed can be easily protected.

The investigation of normal jump, therefore, has been largely made in the experimental field related to the actual engineering problem of spillway design, and even in the present day, before the completion of reservior and diversion projects, the model test is commonly made for the improvement of efficiency of energy dissipation by the hydraulic jump. Furthermore, with respect to jump in sloping channels, B.A. Bakhmeteff and A.E. Matzke[1] and C.E. Kindvater[2] investigated systematically. More recently, J.N. Bradley and A.J. Peterka[3] indicated the well-known relationship was available for the design criteria of practical problems related to spillway, outlet and canal projects, summarizing the tremendous amount of experimental works at the Bureau of Reclamation, U.S.A,

On the other hand, the phenomenon of undular jump can rarely be seen in the literature owing to less importance to practical problems. Interests, however, of the undular jump will be arised when the flow is near critical as in the navigation canal and the outlet work of power canal. Boussinesq, Fawer, R. Lemoine[4], A.M. Binnie and J.C. Orkney[5] and the author[6] studied the hydraulic characteristics of this type of jump. Nevertheless, the final conclusion seems still far away owing to the analytical difficulty as seen in 2-2-1.

The general character of normal jump will be first explained in conclusive forms, as the hydraulic jump which is a typical example of rapidly varied flows has an important relationship to the calculation procedure of surface profiles of gradually varied flows. In the next section, some comments of hydraulic behaviours of undular jump will be described.

2 - 3 - 2 Normal Type of Hydraulic Jump

As has been discussed in the foregoing part, the stationary discontinuity is called as the hydraulic jump by engineers, and Eq.(122) in 1-1-6 thus indicates the relationship between up- and downstream flow characteristics, when the pressure is assumed hydrostatic. The phenomenon of jump is essentially in a steady state, so that the analysis must be derived as the steady state problem. Considering a flow domain involving the jump enclosed two sections, the momentum conservation law in uniform and mild channels is

$$M_o = \int (u^2/g + p/\rho g)dA, \tag{1}$$

and

$$(dM_o/dx) = A\sin\theta - (Q^2/C^2RA). \tag{2}$$

As the change in flow behaviours by the occurrence of jump is practically very sudden, so the change of momentum flux will be assumed extremely small. The basic relation between two sections becomes

$$\int_o (u^2/g + p/\rho g)dA = \int_1 (u^2/g + p/\rho g)dA, \tag{3}$$

in which the subscripts 0 and 1 indicate values at the up- and downstream sections. For the hydraulic analysis of jump, the velocity and pressure distributions at least two sections must be known, so that the classification of a variety of flow behaviours related to the jump becomes of great significance to study. Many investigators have been studied the various types of jump and the conclusive classification of types of hydraulic jump in terms of the Froude number upstream. J.N. Bradley and A.J. Peterka[3)] divided into the following five types of hydraulic jump, with a tremendous amount of systematic experimental observations.

(1) Being the Froude number between 1.0 - 1.7, a slight ruffle on the water surface is the only apparent feature that differentiate this flow from flow at critical depth, and when the Froude number approaches 1.7, a series of smaller waves develops on the surface.

This value of 1.7 is also the condition for undular jumps proposed by Fawer and Bakhmeteff. The surface forms a series of waves whose amplitude decreases downstream, so that the analysis must be included by the curvature of stream tube, and it will be treated in the next section.

(2) Up to a value of 2.5, this action remains much the same, and the phenomenon is called as pre-jump. With respect to the flow characteristics, the velocity throughout is fairly uniform and the energy loss is low.

(3) Between 2.5 and 4.5, the jump has a pulsating action and usually is seen in low head structures. Each oscillation produces a large wave of irregular period that can travel downstream for miles. This jump is classified as transition.

(4) From 4.5 to 9.0, the range is of good jump. The jump is well balanced and the action is thus at its rest.

(5) Over 9.0, the jump becomes violent and the rough surface can prevail in the downstream reach.

The limits of the Froude number given above for the various forms of jump are not definite values, but overlap somewhat depending on local factors. If the analysis only concerns with the change of flow characteristics through the hydraulic jump, the foregoing relation is still effective. Fig. 2-2 indicates

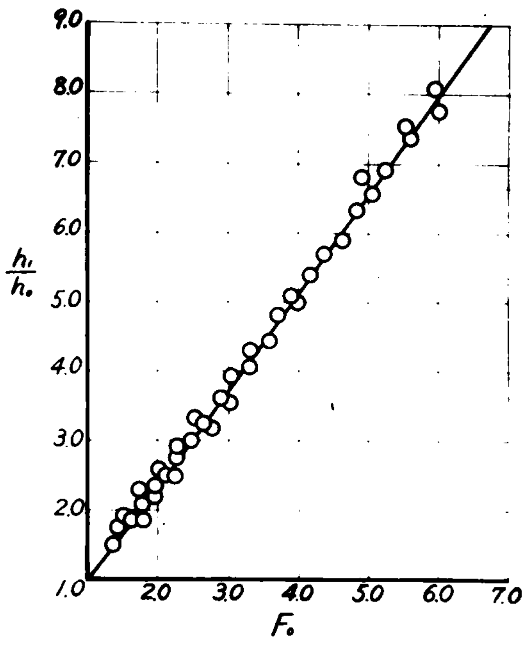

Fig. 2-2 Relationship between up- and downstream depths in terms of upstream Froude number

the relationship between the up-and downstream depths in terms of upstream Froude number, and it is seen the experimental data also represent a good agreement with the theory. In the rapid flow, the pressure will be commonly assumed hydrostatic, as the curvilinear influence of stream tube is not involved, as described in 2-2-1. The momentum relationship of Eq.(3) is then

$$(u_o^2A_o/g)(\alpha_o - \alpha_1 A_o/A_1) = \int_1 p/\rho g dA - y_{Go}A_o\cos\theta. \tag{4}$$

If the hydrostatic pressure also prevails in the downstream flow, the well known relation for all channels shapes is derived, assuming $\alpha_o \doteqdot \alpha_1 \doteqdot 1$,

$$(u_o^2/gy_{Go}\cos\theta) = (A_1y_{G1}/A_oy_{Go} - 1)/(1 - A_o/A_1), \tag{5}$$

and Eq.(5) is the same equation as derived by the momentum theorem for a moving discontinuity. If the channel is of rectangular shape, $y_{Gi} = (h_i/2)$, $A_i = h_ib$, so that Eq.(5) becomes, in terms of the Froude number of $(u_o/\sqrt{gh_o\cos\theta})$,

$$(h_1/h_o) = (-1 + \sqrt{1 + 8F_o^2})/2. \tag{6}$$

This is described in Fig. 2-2. The assumption of hydrostatic pressure in flow is valid in the normal type of flow, because the flow characteristics in pressure is recovered within a short distance through the jump, but the loss of energy is resulted and it is

$$H_o = H_{o1} - H_{oo} = -(h_1 - h_o)^3/4h_oh_1, \tag{7}$$

and a formula of Bresse published in 1860, for hydrostatic flows in rectangular channels. The gain of energy by a discontinuity in flows is impossible, so that the negative jump is not conceivable.

In the one dimensional hydraulics, the jump may be considered as a discontinuous line from shooting to tranquil. Actually, the hydraulic jump involves a surface roller with horizontal axis, and the length to recovery of flow is seen. So far as the foregoing analysis is concerned, the length of jump can not be theoretically evaluated, and many empirical formulas of Smetana[7], Woycicki[8], Bakhmeteff and Matzke[1] and others have been proposed. A large number of experimental data obtained at the Bureau of Reclamation indicate no formulas, being valid to the estimation of length, exist for all ranges of Froude numbers. The measurement of length in hydraulic jump has also been applied to the discharge measurement by some engineers, because the upstream flow is independent of the downstream condition. The estimation of length, however, is most difficult as described, so the possible accuracy can not be expected.

T. Tsubaki[9] studied theoretically the detailed behaviours of hydraulic jump, assuming the velocity profile within the zone of jump was of form of cubic function of the depth. The measurement of velocity and pressure are so difficult owing to a considerable violence involving air bubbles, so that the dynamic explanation of inner physical properties of jump will be remained unsolved before the measuring instruments are developed further.

Of practical importance for hydraulic design of conveyance and control structures is to determine the location of jump. The basic principle of jump is expressible by Eqs.(3) - (6), so both depths are conjugate each other, and therefore, at the point where the upstream depth of flow traced from the upstream control becomes sequent to the downstream depth computed from the downstream control, the hydraulic jump will occur. It is also observed that, if the downstream depth is increased, the jump travels upstream, and conversely if decreased, the jump moves downstream. The illustration of this fact will be seen in the later chapter of this part.

In reality, often occured are the hydraulic jumps on sloping channels such as spillway aprons. Bakhmeteff and Matzke[10] and C.E. Kindsvater[11] investigated the hydraulic behaviours of this type of jump. Recently, Bradley and Peterka compiled a large number of data obtained at the laboratory, USBR, and concluded the expression of Kindsvater was the more common to use.

$$(h_1/h_o) = (1/2\cos\theta)\left[\sqrt{8F_o^2\cos^3\theta/(1 - 2K\tan\theta) + 1} - 1\right], \qquad (8)$$

in which K is the experimental coefficient and θ is the angle between the channel floor and the horizontal. Values of K are also indicated as a function of $\tan\theta$ by Bradley and Peterka and of the Froude number by Hickox[13].

For the design procedure, these charts of Bradley and Peterka are widely available.

2 - 3 - 3 Hydraulic Behaviours of Undular Jump

In the foregoing section, the behaviour of undular jump was briefly described. This problem is frequently encountered in design of canal and spillway transitions. Especially, for low head weir, normal undular jump is observed and for spillway transitions downstream from the crest, oblique undulation also is seen, so that the analysis of this type of jump will serve to the unified treatment and general method of hydraulic design for transition flow from shooting to tranquil in various kinds of hydraulic structures.

Many studies of undular jump have mainly concerned with the limiting condition between undular and normal, and Fawer and Bakhmeteff and Matzke as well as A.T.Ippen and D.R.F. Harleman[14] indicated this condition was 2 in the ratio of down- and upstream depths, by experimental and analytical treatments. Ippen and Harleman used the translation wave theory of Boussinesq-Keulegan-Patterson, expressed in 1-3-3. As already indicated, the theory applied was an approximate solution based on the irrotational gravity waves, and the problem is essentially steady. Possibility, however, of application of the translation wave theory to the present problem is very large, and perhaps it will be adequate. Herewith, the other difficulty of analysis is arised. As the energy loss by the undular jump is so small as to be practically omitted, so the transitional form from shooting to tranquil in surface profiles becomes exclusively of form of solitary wave as a particular case of cnoidal wave. The combined feature of undular jump, which is of gradually increasing pattern in the shooting flow and undular in a form of cnoidal or similar fashion, can not be obtained by solving the basic relationship. When the flow approaches uniform upstream, the only solution is of solitary description, when the vertical influence of appreciable acceleration is applied to the flow, so that the complete feature of undular jump will be approximated by combining two solutions of sub- and supercritical flows together at the initial wave crest. The limiting condition, therefore, is also considered as the breaking condition of waves, though Ippen and Harleman applied the total head concept over a channel bottom to the analysis. Experimental verification also serves to confirmation of basic theory. Nevertheless, the data are not sufficient to verify the theory and more detailed experimentation will be needed.

(a) Surface Profiles of Undular Jump

The usual procedure of jump analysis is based on the momentum approach because of clear description of basic flow patterns, so that the same method will be used for the undular jump. The basic relationship is thus from the results in the foregoing part,

$$gM_0 = (\beta q^2/h) + (gh^2\cos\theta/2) + \gamma q^2\{(d^2h/dx^2) - (1/h)(dh/dx)^2\}. \tag{9}$$

If the uniformity of velocity is assumed, $\beta = 1$ and $\gamma = 1/3$. Assuming the momentum flux is constant throughout the flow zone and the upstream flow is uniform, the surface profile of jump becomes

$$\gamma(dh/dx)^2 = (g\cos\theta/q^2)(h - h_o)^2(\beta q^2/gh_o^2\cos\theta - h), \qquad (10)$$

so that the solution represents a profile of solitary wave expressed by

$$h = h_o + (\beta F_o^2 - 1)h_o \operatorname{sech}^2\{\sqrt{(\beta F_o^2 - 1)/\gamma F_o^2}(x/2h_o)\}. \qquad (11)$$

The maximum wave height of first undulation is apparently $\beta F_o^2 h_o$ and depends on the shooting velocity. The tranquil branch is also expressed as the solution of the following equation

$$\gamma(dh/dx)^2 = -(g\cos\theta/q^2)\{h^3 - (C_1 q^2/g\cos\theta)h^2 + (2M_o/\cos\theta)h - (\beta q^2/g\cos\theta)\}, \qquad (12)$$

in which C_1 is a constant determined by the condition that $dh/dx = 0$ at $h = h_{max} = \beta F_o^2 h_o$. The solution is easily obtainable by integrating Eq.(12) with respect to h and it is

$$h = h_{min} + (h_{max} - h_{min})\operatorname{cn}^2(x/\Delta, k), \qquad (13)$$

in which

$$h_{min} = (2h_o/\beta)\{F_1(\pi/2, k) - E_1(\pi/2, k)\}/[F_1(\pi/2, k)\{\sqrt{1 + 8F_o^2} - 1\} - \beta F_o^2 E_1(\pi/2, k)],$$

$$k^2 = \{\beta F_o^2 - (h_{min}/h_o)\}/\{\beta F_o^2 - (h_o/h_{min})\},$$

and

$$\Delta = 2h_o/\sqrt{(1/\gamma)\{\beta - (h_o/F_o^2 h_{min})\}},$$

and $F_1(\pi/2, k)$ and $E_1(\pi/2, k)$ are the first and second complete elliptic integrals.

The complete configuration of undular waves from shooting to tranquil flows can be presented by connecting two solutions of Eqs. (11) and (13) together at the origin of coordinate system.

(b) Limiting Condition for Undular Jump

As described in the preceding section, the frontal wave of undular jump will become breaking, if the rapid Froude number becomes high. So far as the frontal wave pattern is approximated by the solitary wave, the limiting condition for undular jump will be practically considered as the breaking condition of waves, which, however, has not been established. Usual breaking conditions are determined by the limiting velocity, the limiting shape and the limiting crest angle conditions, as seen in many texts, whereas the definite conclusion is not obtained.

In stationary undular waves, the limiting case to sustain the original pattern will be assumed that the mean velocity at the frontal crest becomes zero, as the reverse current will induce the wave form breaking. The experimental result [15] made at the Hydraulics Laboratory, Kyoto University, verifies the above description. As the shear influence of channel bed will be practically ignored in the undular jump, so therefore, in the following relationship,

$$M_o = (\beta h_o u_o^2/g) + (h_o^2 \cos\theta/2) = (\beta h_{max} u_{max}^2/g) + (h_{max}^2 \cos\theta/2) + (\gamma h_{max}^2 u_{max}^2/g)(d^2h/dx^2)_{x=o}, \tag{14}$$

where dh/dx at the frontal crest is zero and the subscript max indicates the value at the crest, u_{max} becomes zero. With the use of $(h_{max}/h_o) = \beta F_o^2$, the limiting Froude number becomes

$$F_o = 1.553/\sqrt{\beta}\ ,$$

and thus, the maximum depth ratio is

$$(h_{max}/h_o) = 2.414,$$

and the ratio of downstream tranquil depth to upstream shooting depth is 1.754. These results described are less than those of Bakhmeteff and Matzke[1] and Ippen and Harleman[14], who obtained $(h_1/h_o) = 2.000$ and therefore $(h_{max}/h_o) = 2.500$ by the expression of Keulegan for cnoidal waves.

On the other hand, the experimental results of breaking condition for solitary waves on a flat smooth beaches, i = 0 023, conducted by Ippen and G. Kulin[16] indicate (h_{max}/h_o) is about 2.2 in average and those for steep beaches the ratio becomes greater, while the condition obtained by McCowan is 1.78.

If the approximation for the frontal wave is used, the expression becomes of Keulegan's type and the depth ratio is

$$(h_{max}/h_o) \doteqdot (3h_1/2h_o) - (1/2).$$

With the use of this approximation, the same treatment by the momentum approach yields the following relationships

$$F_o = 1.780/\sqrt{\beta}\ ,$$

$$(h_{max}/h_o) = 2.415,$$

and

$$(h_1/h_o) = 1.943,$$

and for the energy approach, they are approximately,

$$F_o = 1.732/\sqrt{\beta}\ ,$$

$$(h_{max}/h_o) = 2.500,$$

and

$$(h_1/h_o) = 2.000.$$

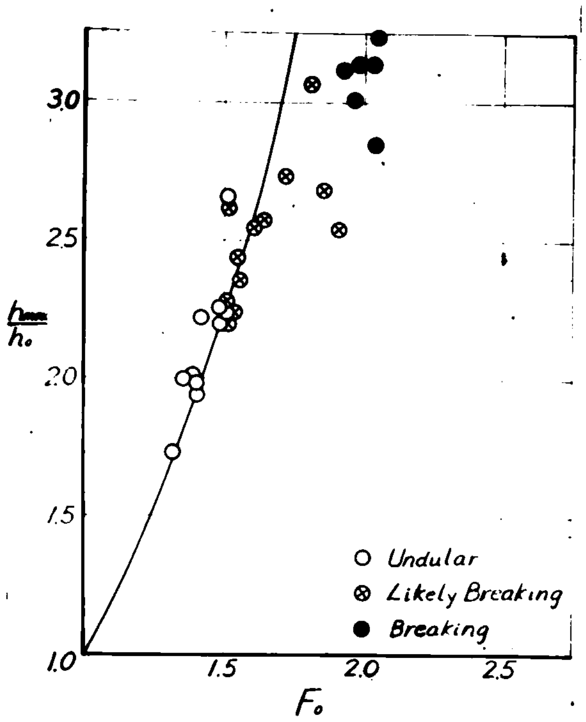

Fig. 2-3 Relationship between (h_{max}/h_o) and F_o

The last results obtained by the energy approach are equivalent to those of Bakhmeteff and Fawer, by their experimental studies, and Ippen and Harleman, by their theoretical and experimental studies for oblique standing waves.

Fig. 2-3 indicates the experimental results of limiting conditions for undular jump obtained at the Hydraulics Laboratory, Kyoto University. Three sorts of marks represent breaking, likely breaking and undular, respectively. The qualification "likely breaking" is that the wave pattern at the frontal and second crests become irregular and breaking. Of special indication for this condition is that the small capillary waves become appreciable in the original surface.

In the figure, the definite conclusion for the breaking conditions described in the foregoing will be not obtained, as the observations were made by eyes. Aside from the theoretical and experimental development on the hydraulics of undular jump, the limiting condition will be practically considered as $F_o = 1.732/\sqrt{\beta}$ and $(h_1/h_o) = 2.000$ for engineering purposes as commonly used.

2 - 3 - 4 Hydraulics of Submergence

In the preceding two sections, the discussion was assumed that the hydraulic jump was occured at a certain location determined by the conjugate relationship between up- and downstream depths. In other words, both depths of up- and downstream from the jump are uniquely determined by the momentum conservation law. If the upstream velocity becomes greater for a particular discharge, the downstream depth of water after passing the jump is also raised, and vice versa.

On the contrary, if the downstream water surface is raised far enough by construction of other hydraulic structures, the influence of downstream depth may be travelled upstream, as understood in the foregoing description on propagation of waves, so that the location of jump will be also transmitted upstream, the hydraulic jump may be submerged until the up- and downstream depths become again conjugated, and the free rapid flow can not be seen. Typical examples of submerged hydraulic jump are seen in the downstream part of free overfalls and stilling pools of overflow spillways as well as in the immediate vicinity of sluice gate.

The hydraulic behaviours of submergence in uniform channels are approximately estimated, if the momentum approach is applied. K. Woycicki [8] evaluated the characteristics of submerged flow under a sluice gate by means of the momentum theorem. However, the detail pattern of submerged flow can not be definitely determined. In reality, the velocity in the upper zone of flow near the free surface is usually upstream, whereas the direction of main flow is downstream, so that the one dimensional procedure for hydraulic analysis is inherently impossible to describe the flow pattern. If the method is used as the engineering approximation, the evaluation of correction coefficients is also impossible. Nevertheless, in gradually varied flows, the influence of submergence may travel far upstream and then the resulting surface profiles are practically estimated with sufficient accuracy.

Usually in the submerged flow, the velocity near the solid boundary is very fast as often observed in the stilling pool, which must be carefully considered when the hydraulic design for stabilization of flow is made. In the case of erodible bed, the jet in the submerged flow carries the sediments up- and downstream, and the results can not be estimated by the theoretical approach. Often caused is the scour downstream from the structure and thus the structure will be possibly imperfect to serve as the stabilization work. The required length of structure must be expanded to a location where the jet is completely diminished.

For hydraulic design procedures of such structures like stilling pools for particular design discharges, the necessary condition is the hydraulic jump of normal type occur in the pool for design discharges, whereas for other values of discharges, the location of jump will move up- and downstream and in some cases the jump will be drowned. Therefore, the design procedure must be carefully made in relation to the occurrence of submerged flows.

---

1) Bakhmeteff, B.A., and Matzke, A.E., The Hydraulic Jump in Sloped Channels, Trans. ASCE, Vol. 60, 1938.

2) Kindsvater, C.E., The Hydraulic Jump in Sloping Channels, Proc. ASCE, Jan. 1949.

3) Bradley, J.N., and Peterka, A.J., The Hydraulic Design of Stilling Basin: Hydraulic Jump on a Horizontal Apron (Basin I), Stilling Basin with Sloping Apron (Basin V), Jour. Hydraulics Division, Proc. ASCE, Oct. 1957.

4) Lemoine, R., Sur les ondes positives dans les canaux et sur le ressaut ondule, La Houille Blanche, Vol. 3, No. 2, 1948.

5) Binnie, A.M., and Orkney, J.C., The Formation of Hydraulic Jump, Proc. Roy. Soc. A., 230, 1955.

6) Iwasa, Y., Undular Jump and Its Limiting Condition for Existence, Proc. 5 th Japan Nat. Cong. Appl. Mech., 1955.

7) Smetana, Inland-Water Navigation Congress, Brussels, 1935.

8) Woycicki, K., Wassersprung, Deckwalze und Ausfluss einer Schutze, Warsaw, 1931.

9) Tsubaki, T., Theory of Hydraulic Jump, Repors, Res. Inst. for Fluid Eng., Kyushu University, Vol. 6, No. 2, 1950.

10) Bakhmeteff, B.A., and Matzke, A.E., The Hydraulic Jump in Terms of Dynamic Similarity, Trans. ASCE, Vol. 101, 1936.

11) Kindsvater, C.E., The Hydraulic Jump in Sloping Channels, Trans. ASCE, Vol. 109, 1944.

12) Bradley, J.N., and Peterka, A.J., Hydraulic Design of Stilling Basin: Stilling Basin with Sloping Apron (Basin V), Jour. Hydraulics Division, Proc. ASCE, HY 5, Oct. 1957.

13) Hickox, Discussion of "The Hydraulic Jump in Sloping Channels", by C.E. Kindsvater, Trans. ASCE, Vol. 109, 1944.

14) Ippen, A.T., and Harleman, D.R.F., Verification of Theory for Oblique Standing Waves, Proc. ASCE, Vol. 80, Separate 526, Oct. 1954.

15) Ippen, A.T., and Kulin, G., Shoaling and Breaking Characteristics of the Solitary Wave, Technical Report No. 15, Hydrodynamics Laboratory, MIT, Apr. 1955.

4. Transitional Characteristics of Gradually Varied Flows and Their Surface Profiles of Water in Channel Transitions and Controls

2 - 4 - 1 Basic Remarks

Surface profiles of water for gradually varied flows in non-uniform channels, such as natural channels and artificial water-courses with frequent changes of channel geometry, grade, boundary materials and so on, can not be predicted by the analytical methods developed by many hydraulic engineers for long time. Usual procedures to makd calculation of surface profiles are numerical or graphical methods, which have also been devised for various purposes. These procedures, however, frequently imply much errors for calculation, if the transitional characteristics described in the foregoing are not evident before calculation.

In this chapter, the transitional characteristics of gradually varied flows will be concerned for the purpose of clear formulation of general characters of steady flow theory since Bresse as the extensive theory. The gradually varied flow is also characterized by the hydrostatic pressure law, which makes the analytical method by the momentum approach in a very convenient form, owing to the complexity for the expression of the frictional resistance by the energy approach.

The application of the geometric theory of ordinary differential equation to the open channel flow was initiated by P. Massé[1)] in 1938. His subject was to reveal a thorough investigation of theory of Bresse as applied to the case of variable slope which passes continuously from steep to mild or vice versa. The problem treated was rather simple, but the basic philosophy implied in his treatment will be of great significance in the theory of steady flows.

After his work, in 1956, F.F. Escoffier[2] investigated the same transitional behaviours of gradually varied flows for his intention of application to the graphical method of surface profile calculation, without establishing the definite characteristics of channel transitions. More recently, in 1958, the author[3),4)] concerned with the same problems of flows in channels with non-uniformity in channel geometry, and investigated thoroughly the transitional characteristics contributive to complete procedure to trace the surface profile and the distinct determination of reservoir capacity.

This chapter first deals with the hydraulic characters of singular points as the transitional points from tranquil to rapid or vice versa and their hydraulic significance, followed by the further comment to Massé's theory. The next is to concern with the transitional characteristics of Chézy flows, as one of the most important expression for open channel flows. The transitional characteristics in channel transitions and controls are distinctively classified in terms of channel geometry and surface resistance. Each of these behaviours in transition flows is substantially significant to determine the surface profiles of water, and therefore the formulation of transitional characteristics provides the guiding for hydraulic design of conveyance and control structures. The last section describes some examples of surface profiles observed in natural channels and watercourses. It is apparently observed various types of surface profiles will be occured under various conditions of channel characteristics.

As typical examples of application of the theory described herein to the actual hydraulic works, various types of flumes for discharge metering by a single water-level measurement will be seen. Flumes of Parshall, Inglis, de Marchi and others are so familiar to hydraulic engineers. The theory and hydraulic performance of flume structures will be illustrated in the next part, as the functionality of flume is commonly classified as control structures.

2 - 4 - 2 Singular Points of Gradually Varied Flows in Channel Transitions and Controls and Their Hydraulic Significance to Transitional Behaviours

(a) Location of Singular Point

As before explained, the momentum method in the one dimensional approach of open channel flows will be applied to the analysis, so that the surface profile equation* is then expressed as, from Eq.(72) in 1-1-4, under the assumption that the local curvature is extremely small and β is constant and nearly unity for any section of flow,

$$(dh/dx) = \sin\theta - (\tau/\rho gR) + (\beta Q^2/gA^3)(\partial A/\partial x) /\{\cos\theta - (\beta Q^2/gA^3)(\partial A/\partial h)\}. \quad (1)$$

When the Chézy law for boundary resistance will be used to Eq.(1), but the Chézy roughness is a variable, Eq.(1) becomes

$$(dh/dx) = \{\sin\theta - (Q^2/C^2RA^2) + (\beta Q^2/gA^3)(\partial A/\partial x)\}/\{\cos\theta - (\beta Q^2/gA^3)(\partial A/\partial h)\}. \quad (2)$$

If the channel is of rectangular shape, $A = bh$ and thus $(\partial A/\partial x) = h(db/dx)$ and $(\partial A/\partial h) = b$.

The critical condition in which the denominator becomes zero is

$$\cos\theta_c = (\beta Q^2/gA_c^3)(\partial A/\partial h)_c,$$

and the curve of normal depth for a particular discharge and definite channel is

$$\sin\theta_c = (Q^2/C_c^2R_cA_c^2) - (\beta Q^2/gA_c^3)(\partial A/\partial x)_c,$$

and then the bed slope at the singular point is

$$\tan\theta_c = i_c = (gA_c/\beta C_c^2R_c)/(\partial A/\partial h)_c - (\partial A/\partial x)_c/(\partial A/\partial h)_c. \quad (3)$$

*** The term depending on the change of bottom grade is of small magnitude, and it is practically ignored.**

Introducing the local critical slope i_{cr}, which is only defined for the flow in uniform channels

$$i_{cr} = (gA_c/\rho C_c^2 R_c)/(\partial A/\partial h)_c, \tag{4}$$

and inserting the expression for i_{cr} into Eq.(3), the relationship between the local grade and the channel geometry at the singular point is obtained in the following.

$$(\partial A/\partial x)_c/(\partial A/\partial h)_c = i_{cr} - i_c. \tag{5}$$

This equation is an important relationship for the existence of singular point in the basic flow of gradually varied regime and is characterized as follows:

As the flow area in channels is an increasing function of the depth, so $(\partial A/\partial h)$ is positive for any channel geometry.

(1) In divergent channels, $(\partial A/\partial x)_c > 0$, a singular point will be appeared at a section where the channel slope is mild, $i_{cr} - i_c > 0$,

(2) On the contrary, convergent channels, $(\partial A/\partial x)_c < 0$, produces a singular point in a channel reach of steep slope, $i_{cr} - i_c < 0$,

(3) If the channel is uniform in shape, $(\partial A/\partial x)_c$ becomes zero, so that a singular point will be observed at a section where the channel grade becomes equal to the local critical slope determined by the given discharge.

(b) Classification of Singular Points

As the location of singular point is determined from the consideration in (a), so the origin of coordinates will be transfered to the point. Let denote the point by x_c, h_c, ··· . The linearized version of Eq.(2), which describes the approximate behaviours of surface profile of gradually varied flows, becomes

$$(dh/dx) = (cx + dh)/(ax + bh), \tag{6}$$

in which

$$a = -\sin\theta_c(d\theta/dx)_c + (3\beta Q^2/gA_c^4)(\partial A/\partial x)_c(\partial A/\partial h)_c - (\beta Q^2/gA_c^3)(\partial^2 A/\partial x\partial h)_c,$$

$$b = (3\beta Q^2/gA_c^4)(\partial A/\partial h)_c^2 - (\beta Q^2/gA_c^3)(\partial^2 A/\partial h^2)_c,$$

$$c = \cos\theta_c(d\theta/dx)_c + (2Q^2/C_c^3R_cA_c^2)(\partial C/\partial x)_c - (Q^2/C_c^2R_c^2A_c^2)(\partial R/\partial x)_c + (2Q^2/C_c^2R_cA_c^3)(\partial A/\partial x)_c - (3\beta Q^2/gA_c^4)(\partial A/\partial x)_c^2 + (\beta Q^2/gA_c^3)(\partial^2 A/\partial x^2)_c,$$

$$d = (2Q^2/C_c^3R_cA_c^2)(\partial C/\partial h)_c + (Q^2/C_c^2R_c^2A_c^2)(\partial R/\partial h)_c + (2Q^2/C_c^2R_cA_c^3)(\partial A/\partial h)_c - (3\beta Q^2/gA_c^4)(\partial A/\partial x)_c(\partial A/\partial h)_c + (\beta Q^2/gA_c^3)(\partial^2 A/\partial x\partial h).$$

and the subscript denotes the critical state.

The classification of singular points, saddle, nodal and focal, is divided by the mathematical property of characteristic equation of (6), as already indicated in 1-2-4, and therefore the transitional characteristics are also evaluated. Its treatment of gradually varied flow for any characteristic in channel geometry and boundary is of great complexity, whereas Eq.(6) is derived without assumptions in the basic physics of gradually varied flow but β = const, so that the exact estimation of surface profiles of gradually varied flows in natural channels with frequent changes in channel characteristics must be followed by Eq.(6).

The general character of transitional behaviours of gradually varied flows can not be explicitely explained, and in all cases encountered in the actual problems of hydraulic design, the possible procedure is only the numerical analysis, so that the clarification of transitional points and transitional behaviours through the point is of primary importance before calculation. Some particular cases in which the flow is assumed of Chézy or Manning type in rectangular channels will be treated in the later section.

(c) Some Comments to Theory of Massé

As briefly described, Massé[1)] first applied the geometric theory of ordinary differential equation to the Bresse flow, which is classified as the Chézy flow in uniform rectangular channels in the present study, in open channels of variable slope which passes continuously from steep to mild or vice versa, as overflow spillways and concluded three types of transitions would be possible. The most significant results are that the smooth transition from rapid to tranquil without producing the hydraulic jump will be possible under certain conditions. Although the details of his procedure can not be seen owing to difficulty to obtain his original paper, the present procedure will be readily applied to the theory of Massé as one of the most simplified extention of Bresse curve of surface profiles.

In the Bresse flow in rectangular channels, $Q = qb$, $A = bh$, $C = \text{const.}$, and $\beta = 1$, where b is the width of channel, so that coefficients of a, b, c and d in Eq.(6) become , in the immediate vicinity of singular point located at a section where the local slope becomes critical, $i_{cr} = (g/C_c^2)$.

$$a = -\sin\theta_c(d\theta/dx)_c,$$
$$b = 3\cos\theta_c/h_c,$$
$$c = \cos\theta_c(d\theta/dx)_c,$$
$$d = 3\sin\theta_c/h_c,$$

with the aid of $\sin\theta_c = (q^2/C_c^2h_c^3)$ and $\cos\theta_c = (q^2/gh_c^3)$.

The characteristic equation is then

$$S^2 - \sin\theta_c\{(3/h_c) - (d\theta/dx)_c\}S - (3/h_c)(d\theta/dx)_c = 0, \qquad (7)$$

so that a singular point becomes focal if the discriminant is negative and saddle or nodal for positive values of D. The distinction between focal and saddle or nodal is determined by

$$(h_c/3)(d\theta/dx)_c > -(1 - \cos\theta_c)/(1 + \cos\theta_c), \text{ and}$$

$(h_c/3)(d\theta/dx)_c < -(1 + \cos\theta_c)/(1 - \cos\theta_c)$,

for saddle and nodal points,

$-(1 - \cos\theta_c)/(1 + \cos\theta_c) > (h_c/3)(d\theta/dx)_c > -(1 + \cos\theta_c)/(1 - \cos\theta_c)$,

for focal point.

On the other hand, the sign of product of two roots indicates the discrimination of singular point saddle or nodal, and it is under $D > 0$,

$(d\theta/dx)_c > 0, \cdots\cdots$ saddle, and $(d\theta/dx)_c < 0, \cdots\cdots$ nodal .

The transitional behaviours of Bresse flow, therefore, will be explained in the following.

The singular point is produced at a section of critical slope.

(1) When the grade of channel bottom increases gradually as observed at the downstream side of overflow spillway crest, a saddle point is seen and the flow changes its regime from tranquil to rapid. The curve of transition profile is evidently the Cl-curve defined in 1-2-4, as b is positive, and this point becomes a channel control. The transition slope is

$$(dh/dx)_{c1} = (i_{cr}/6)\left\{h_c(d\theta/dx)_c + 3 - h_c\sqrt{(d\theta/dx)_c^2 + (6/h_c)(1 + \cos^2\theta_c)(1 - \cos^2\theta_c)^{-1}(d\ \ /dx)_c + (9/h_c^2)}\right\}. \qquad (8)$$

and $(a - d) < 0$, $b > 0$, and $c > 0$, so that the above expression is of negative value.

(2) When the channel grade decreases gradually as seen near the spillway sill, nodal or focal point is produced in the flow. For slightly and rapidly decreasing slopes, the flow changes from rapid to tranquil through the nodal point without or near the point with accompanying hydraulic jump determined by the downstream flow condition, while for moderate decreasing channels, the flow changes by the hydraulic jump near the focal point.

The above conclusion is different from that of Massé cited in the literature of Jaeger[5], though the original paper is not seen. The reason of this discrepancy is explained by the fact that Massé used the slope of transition curve at the singular point was expressed as

$$(dh/dx)_c = -(h_c/3i_c)(d\theta/dx)_c.$$

Evidently, this value is not the slope of transition curve but that of the normal depth curve and both values are not coincided, so that his theoretical deduction was not valid. Fig. 2-4 indicates the classification of transitional behaviours through various singular points to the channel grade.

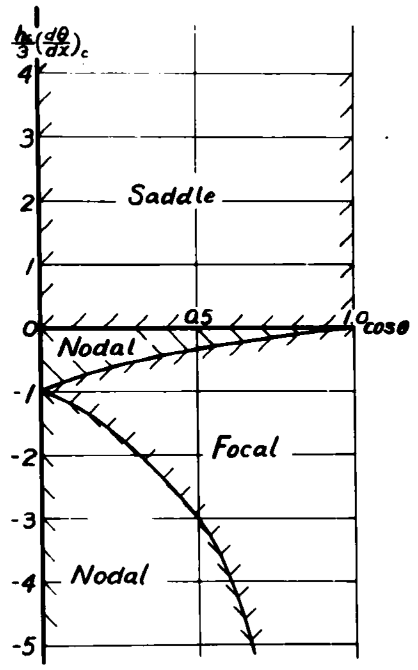

Fig. 2-4 Classification of transitional behaviours of Bresse flow in channels of variable slopes to channel grade

2 - 4 - 3 Hydraulic Characteristics of Transitional Behaviours of Chézy Flows

(a) Classification of Transitional Behaviours of Chézy Flows

The foregoing section concerned with the most simplified case in transitional behaviours studied by Massé. The investigation of Chézy flows was initiated by Bresse as far back as 1860, and thereafter many hydraulic engineers studied the problem in refined form. The study, however, is limited only to the classification of all the possible surface profiles resulted from the local change of water elevation in uniform channels. Massé extended the Bresse theory to the flow in variable slopes and M. Homma[6], also in 1955, to the flow in non-uniform channels with constant grade and broad width, by means of rather empirical procedures without rigorous mathematical knowledge. In 1958, the author completed the classification of transitional behaviours of Chézy flow in rectangular channels of constant grade with broad width[3] and treated in a more general case[4].

In this section, the details of transitional behaviours of Chézy flow, which is a basis for the hydraulics of gradually varied flows in open channels, in rectangular channels with variable gr de and width will be explained.

Coefficients of a, b, c and d are

$$a = -i_{cr}\{\alpha p + 2(\alpha - 1)\}\{\beta - 2(\alpha - 1)\},$$

$$b = 3\{\beta - 2(\alpha - 1)\},$$

$$c = i_{cr}^2[(p/i_{cr}^2)\{\beta - 2(\alpha - 1)\} + 6\alpha(\alpha - 1)^2 - \beta(\alpha - 1)(3\alpha - 1) + (m/3)\{\beta - 2(\alpha - 1)\}],$$

$$d = i_{cr}\{\beta(2\alpha + 1) - 4\alpha(\alpha - 1)\},$$

in which α; i_c/i_{cr}, β : $(db/dx)_c/i_{cr}$, m: $3h_c^2(d^2b/dx^2)_c/b_c i_{cr}^2$, and p: $h_c(d\theta/dx)_c$, and they indicate dimensionless channel grade, change of width, change of width curvature and change of grade, respectively.

The first attention is directed to the relationship between a, b, c and d, and the channel geometry in shape and grade.

If the $\alpha - \beta$ plane is chosen for this purpose, the behaviour of a is influenced by the dimensionless grade change, and the regions for existence of singular points are divided into two parts: one is the region where the channel is steep and convergent and the other the channel is mild or adverse and divergent

The expression for a is again

$$a = -2i_{cr}(1 + p/2)\{\alpha - 2/(2 + p)\}\{\beta - 2(\alpha - 1)\}, \tag{9}$$

so that the influence of the parameter p divides the behaviour of a into the following three ways, with the aid of the figure, which indicates the relationship between the dimensionless grade, i_c/i_{cr}, and the dimensionless grade change, $h_c(d\theta/dx)_c$.

(1) $p \geqq 0$. The root of α in the second bracket of Eq.(9) is between 1 and 0. The sign of a is positive for steep convergent and also adverse channels, while for mild channels a becomes positive for very mild slopes between 0 and $\alpha_1 = 1/(1 + p/2)$ and negative for mild slopes between α_1 and the critical slope. If the channel grade is constant, so that p = 0 and a is positive for any characteristic in channel slope as described in the paper of the author[3].

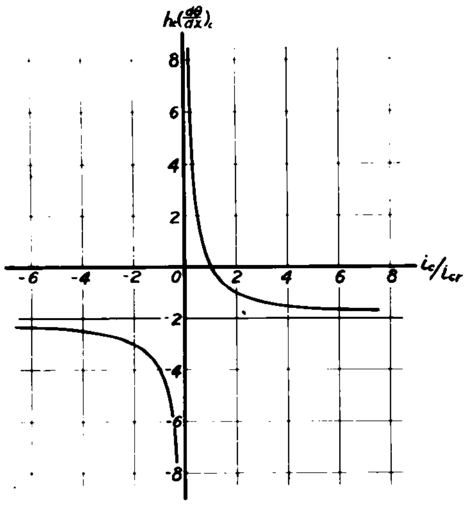

Fig. 2-5 Relationship between α_1 and p for making a zero

(2) $0 > p > -2$. α which makes a zero is between 1 and ∞. In mild and adverse slopes a is definitely positive, while a is positive or negative, depending on the values of dimensionless grade, in steep slopes.

(3) $p < -2$. As α which makes a zero is in the negative zone of abscissa, so a is positive in mild and steep channels. On the contrary, in adverse slopes a becomes positive or negative.

The behaviour of b is readily understood and positive in divergent channels, and thus mild and adverse in channel grade, while in convergent channels b becomes negative.

The expression for d is transformed into

$$d = i_{cr}(2\alpha + 1)\{\beta - 4\alpha(\alpha - 1)/(2\alpha + 1)\}, \tag{10}$$

and evidently, $\alpha = -0.5$ is a barrier of change in sign. The upper and lower planes divided by $\beta = 4\alpha(\alpha - 1)/(2\alpha + 1)$ become the positive and negative regions, corresponding with $\alpha \gtrless -0.5$. Or hydraulically speaking, d is negative for convergent and steeply adverse slopes, and on the contrary, d is positive for mild slopes and gently adverse slopes with extremely large divergence in channel width.

With regard to the behaviour of c, which is a function of α, β, m and p, it must be indicated in terms of two parametric expressions of m and p in $\alpha - \beta$ plane. The expression of c is

$$c = -3i_{cr}^2\{(\alpha - 2/3)^2 - (1 + m + 3p/i_{cr}^2)/9\}[\beta - 2(\alpha - 1)\{(\alpha - 1/2)^2 - (9 + 4m + 12p/i_{cr}^2)/36\}/\{(\alpha - 2/3)^2 - (1 + m + 3p/i_{cr}^2)/9\}], \qquad (11)$$

so that the general characters of c can not be explicitely described. If the change of grade becomes zero and thus the channel is of variable shape in cross section, the behaviour of c is readily classified in the following.

(1) $m \leq -2.25$.

$$c = -3i_{cr}^2 M\{\beta - (2N/M)(\alpha - 1)\},$$

where

$$M = (\alpha - 2/3)^2 - (1 - m)/9 > 0$$

$$N = (\alpha - 1/2)^2 - (9 - 4m)/36 \geq 0$$

and consequently, c is negative for divergent channels and positive for convergent channels.

(2) $-1 > m > -2.25$.

$$c = -3i_{cr}^2 M\{\beta - 2(\alpha - 1)(\alpha - \alpha_3)(\alpha - \alpha_4)/M\},$$

in which

$$\alpha_{3,4} = \{1 \pm \sqrt{1 + (4m/9)}\}/2.$$

For the range of $-1 > m > -2.25$, the magnitude of α_i is arranged as $\alpha = 1 > \alpha_3 > \alpha_4$, and thus c is positive in steep channels and mild channels in which β is of small value between α_3 and α_4 and negative for adverse slopes and mild slopes in which the channel geometry at the transitional point is outside of the foregoing region, as evidently seen in Fig. 2-6.

(3) $0 > m \geq -1$

As $1 > \alpha_3 > \alpha_1 \geq \alpha_2 > \alpha_4 > 0$ for this range, so the behaviour of c in sign is indicated in Fig. 2-7. Evidently it is seen that the positve region of c for mild channels increases gradually with the increase of m.

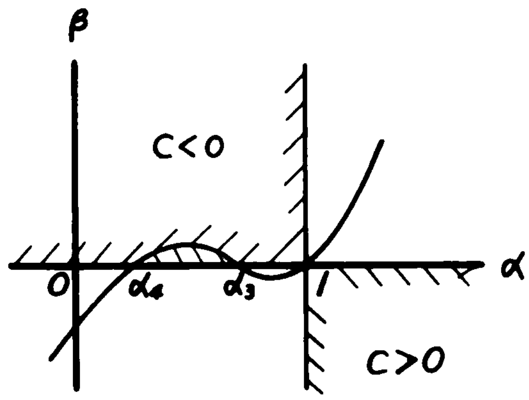

Fig. 2-6 Behaviour of channel characteristics to sign of c in Chezy flows $(-1 > m > -2.25)$

(4) $m \geq 0$

For all channels of steep, mild and adverse, the sign of c indicates positive or negative depending on the channel geometry, and especially in the case of $m > 3$, c is definitely positive in mild channels with any characteristic in channel geometry. Fig. 2-8 indicates the sign of c for $m \geq 0$.

Figs. 2-9 and -10 indicate some examples of behaviours of channel characteristics to a, b, c and d for the Chézy flow in channels of constant grade, and the details are seen in the study of the author[3)].

The classification of singular points, therefore, will be derived by the property of characteristic equation of

$$S^2 - i_{cr}\{\beta(3 - \alpha p) - 2(\alpha - 1)(2 - \alpha p)\}S + i_{cr}{}^2\{\beta - 2(\alpha - 1)\}[\beta\{5(\alpha - 1)^2 - p(2\alpha^2 + \alpha + 3/i_{cr}{}^2) - m\} - 2(\alpha - 1)\{5\alpha(\alpha - 1) - p(2\alpha^2 + 3/i_{cr}{}^2) - m\}] = 0. \qquad (12)$$

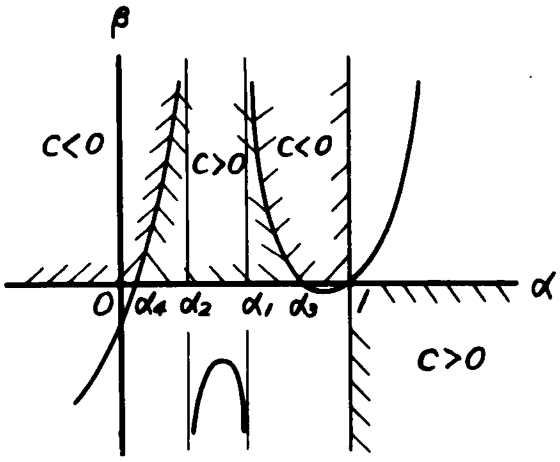

Fig. 2-7 Behaviour of channel characteristics to sign of c in Chezy flows ($0 > m > -1$)

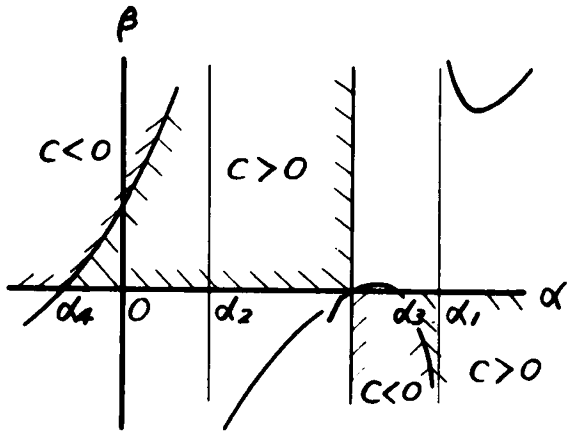

Fig. 2-8 Behaviour of channel characteristics to sign of c in Chezy flows ($m \geq 0$)

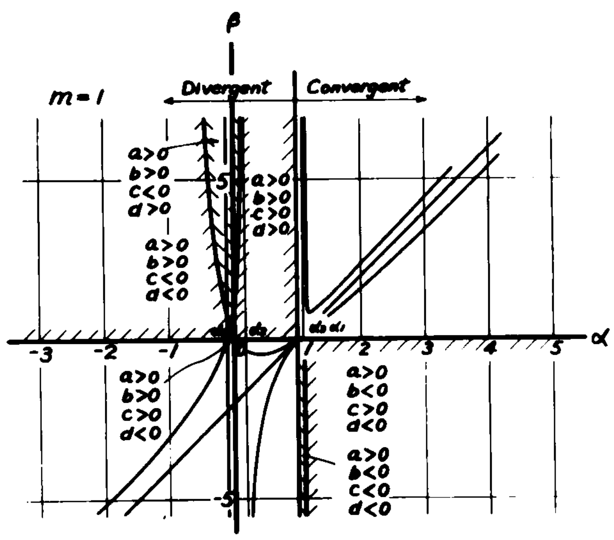

Fig. 2-9 Behaviours of boundary characteristics to a, b, c and d for Chézy flows ($m = 1$)

If the discriminant of the above equation is positive, the singular point is saddle or nodal, while the negative discriminant makes the singular point focal. The discrimination between saddle and nodal depends on the sign of product of two roots, and therefore

$$i_{cr}^2\{\beta - 2(\alpha - 1)\}\Big[\beta\{5(\alpha - 1)^2 - p(2\alpha^2 + \alpha + 3/i_{cr}^2) - m\}$$
$$-2(\alpha - 1)\{5\alpha(\alpha - 1) - p(2\alpha^2 + 3/i_{cr}^2) - m\}\Big] \gtrless 0, \qquad (13)$$

where the upper and lower inequalities indicate the point is classified as nodal and saddle.

As examples of classifications of singular points of gradually varied flows, a simple case in which the channel grade is constant will be treated.

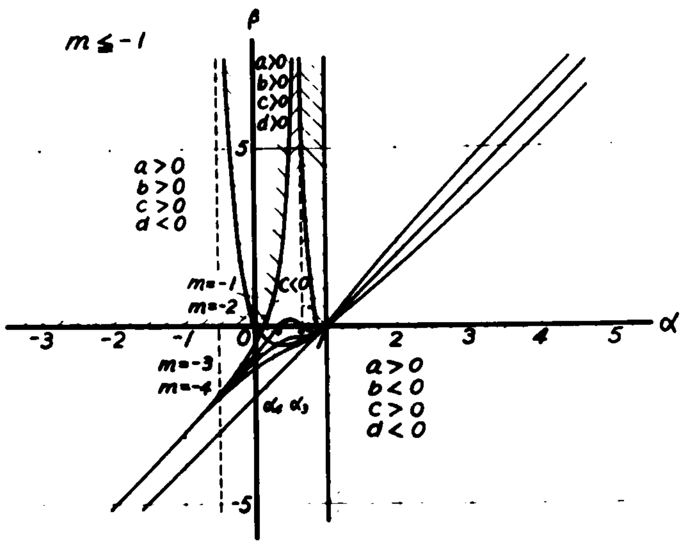

Fig. 2-10 Behaviours of boundary characteristics to a, b, c and d for Chézy flows ($m \leq -1$)

As p is zero, so the discriminant is

$$D = -20i_{cr}^2\{(\alpha - 1)^2 - (9 + 4m)/20\}\Big[\big[\beta - (\alpha - 1)(10\alpha^2 - 15\alpha + 2 - 2m)/5\{(\alpha - 1)^2 - (9 + 4m)/20\}\big]^2 - 24(\alpha - 1)^2\{\alpha - (3/4) - (1/4)\sqrt{1 + (8m/15)}\}\{\alpha - (3/4) + (1/4)\sqrt{1 + (8m/15)}\}/\{(\alpha - 1)^2 - (9 + 4m)/20\}\Big] \gtrless 0. \quad (14)$$

The classification of positive and negative regions thus is in the following.

(1) $m < -2.25$

The first bracket of Eq.(5) is positive, and consequently the inner domain enclosed by $\beta_1(\alpha, m)$ and $\beta_2(\alpha, m)$ curves where

$$\beta_{1,2} = \frac{(\alpha - 1)\ (10\alpha^2 - 15\alpha + 2 - 2m) \pm \sqrt{30\{(\alpha - 3/4)^2 - (15 + 8m)/240\}}}{}\Big]/5\{(\alpha - 1)^2 - (4m + 9)/20\},$$

makes the discriminant positive and therefore the transitional point is nodal or saddle, whereas the outside domain yields the focal point.

(2) $-1.875 > m \geq -2.25$

For this range of m, the discriminant becomes

$$D = -20i_{cr}(\alpha - \alpha_1)(\alpha - \alpha_2)(\beta - \beta_1)(\beta - \beta_2),$$

and for $\alpha > \alpha_1 = 1 + \sqrt{(9 + 4m)/20}$ and $\alpha < \alpha_2 = 1 - \sqrt{(9 + 4m)/20}$, the same conclusion is obtained as in the case of (a). On the contrary, for $\alpha_1 > \alpha > \alpha_2$, D becomes positive and negative in the outer and inner domaind between β_1 and β_2 curves, respectively.

(3) $m \gtrless -1.875$

The second bracket of Eq.(5) has two roots of, for $\alpha \geqq \alpha_1$ and $\alpha \leqq \alpha_2$,

$$\beta_{3,4} = (\alpha - 1)\{(10\alpha^2 - 15\alpha + 2 - 2m) \pm \sqrt{30(\alpha - \alpha_3)(\alpha - \alpha_4)}\ /5(\alpha - \alpha_1)(\alpha - \alpha_2),$$

in which $\alpha_{3,4} = \{3 \pm \sqrt{1 + (8m/15)}\}/4$.

As $\alpha_1 > \alpha_3 > \alpha_4 > \alpha_2$ for this range of m, so for $\alpha > \alpha_1$ and $\alpha < \alpha_2$, the inner domain makes D positive and also D is positive in the outer domain for $\alpha_4 > \alpha > \alpha_2$ and $\alpha_1 > \alpha > \alpha_3$. On the other hand, for $\alpha_3 \geqq \alpha \geqq \alpha_4$, the first bracket is negative and the second one positive without the change of diversity in channel geometry.

The discrimination between saddle and nodal is obtained by the sign of

$$i_{cr}^2\{\beta - 2(\alpha - 1)\}[5(\alpha - 1)^2(\beta - 2\alpha) - m\{\beta - 2(\alpha - 1)\}] \gtrless 0, \qquad (15)$$

in which the lower and upper inequalities indicate the condition for saddle and nodal, respectively. Therefore,

(1) $m \geqq 0$. The transitional point is classified as a saddle point in the inner domain enclosed by two curves of $\beta = 2(\alpha - 1)$ and $\beta = 2(\alpha - 1)(\alpha^2 - \alpha - m/5)/\{\alpha - (1 + \sqrt{m/5})\}\{\alpha - (1 - \sqrt{m/5})\}$ for the range of $\alpha > (1 + \sqrt{m/5})$ and $\alpha < (1 - \sqrt{m/5})$ and the outer domain for $(1 + \sqrt{m/5}) > \alpha > (1 - \sqrt{m/5})$. The other domain makes the point nodal.

(2) For negative values of m, the first bracket of Eq.(6) is positive and the inner and outer domains represent the condition for saddle and nodal, respectively.

Following the foregoing conclusion for the classification of transitional points in channel transitions and controls, the relationship between the classifications and the channel and flow characteristics in channels are obtained, and Figs. 2-11 and -12 describe examples of the hydraulic behaviours of boundary characteristics to transitional characteristics for Chézy flows.

In examples illustrated herein, the bed slope was assumed constant. Actually, it is variable, so that the exact evaluation of transitional behaviours is essentially needed to obtain the desirable solution for hydraulic design.

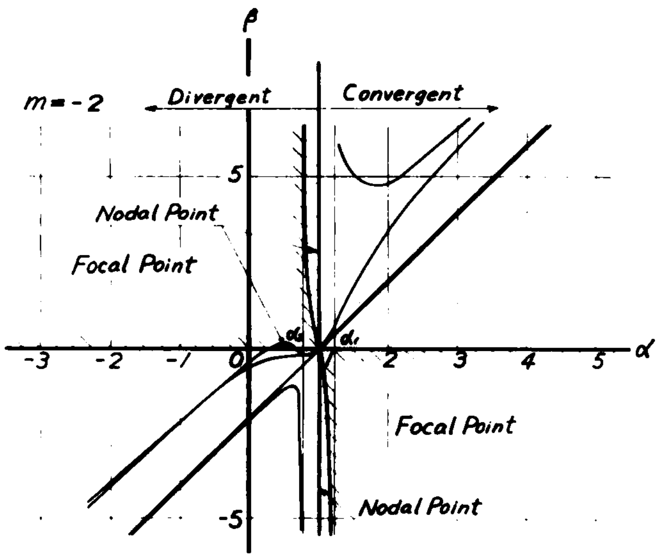

Fig. 2-11 Hydraulic behaviours of channel and flow characteristics to transitional characteristics for Chézy flows (m = -2)

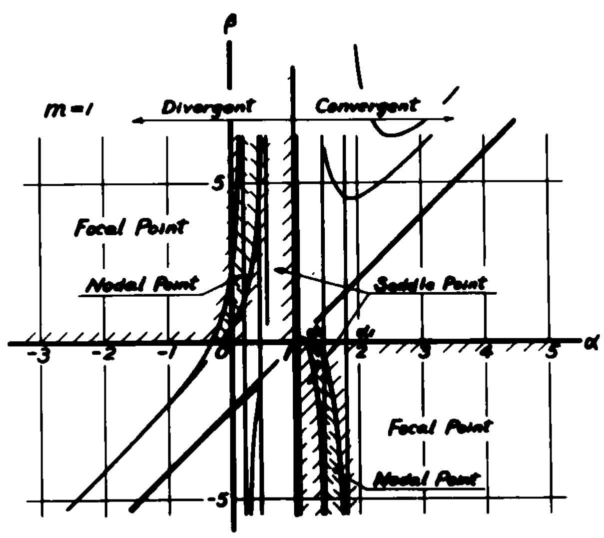

Fig. 2-12 Hydraulic behaviours of channel and flow characteristics to transitional characteristics for Chézy flows (m = 1)

(b) Transitional Characteristics and Surface Profiles of Chézy Flows

As the conclusive characteristics of transitional behaviours of

Chézy flows near the singular point, the following tabulation and figures of surface profiles of water are obtained by the combined features of geometric characters in channel transitions and controls.

(1) Transitional characteristics of saddle point

As has been described in 1-2-4, the transition flow changes its regime from tranquil to shooting and curves except singular solutions indicate surface profiles traced under certain conditions. The characteristics of transitional behaviours are provided by coefficients of a, b, c and d as well as negative S_1 and positive S_2 determined by particular flow and channel characteristics. The signs of a and c in all channel grade of steep, mild and adverse are positive or negative. On the other hand, b is negative in steep slopes and positive in mild and adverse slopes. The sign of d is negative in steep channels and positive in mild channels, whereas positive or negative in adverse channels depending on the characteristics of channel and flow.

The following tabulation indicates the transitional characteristics of Chézy flows through the saddle point and Fig. 2- 13 (1) - (9) also describe classifications of surface profiles near the saddle point. It is seen for the Chézy flow that nine types of transitional behaviours are produced by the saddle point.

Transitional Characteristics of Chézy Flows through Saddle Point

Channel grade	Channel characteristics						Transition curve	Figure	Remarks
	a	b	c	d	s_1	s_2			
Steep	+	-	+	-	+	+	C2	(1)	

	+	-	-	-	-	+	C2	(2)	
	-	-	+	-	+	-			$ad - bc \nless 0$
	-	-	-	-	-	-	"	(3)	
Mild	+	+	+	+	-	-	C1	(4)	
	+	+	-	+	+	-			$ad - bc \nless 0$
	-	+	+	+	-	+	"	(5)	
	-	+	-	+	+	+	"	(6)	
Adverse	+	+	+	+	-	-	C1	(4)	
	+	+	+	-	+	-	"	(7)	
	+	+	-	+	+	-			$ad - bc \nless 0$
	+	+	-	-	-	-	"	(8)	
	-	+	+	+	-	+	"	(5)	
	-	+	+	-	+	+	"	(9)	
	-	+	-	+	+	+	"	(6)	
	-	+	-	-	-	+			$ad - bc \nless 0$

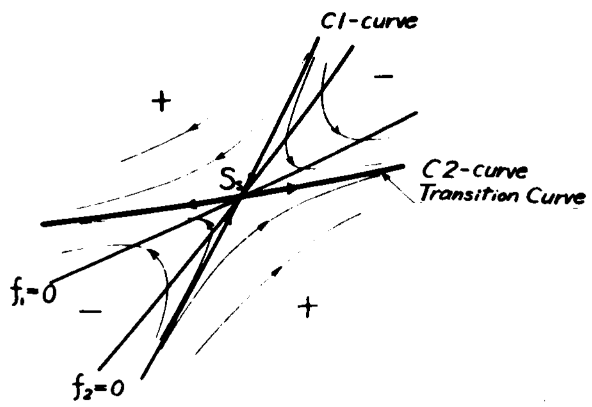

Fig. 2-13 (1) Transitional behaviours of flows through saddle point in steep slopes

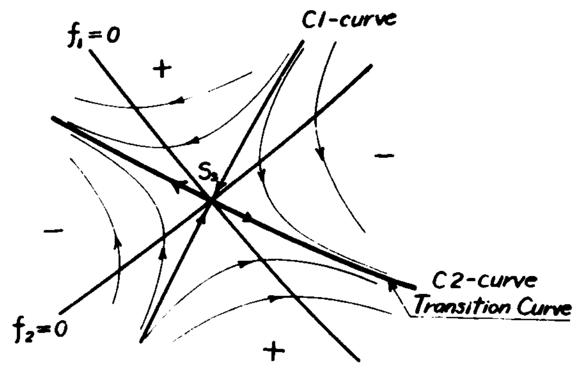

Fig. 2-13 (2) Transitional behaviours of flows through saddle point in steep slopes

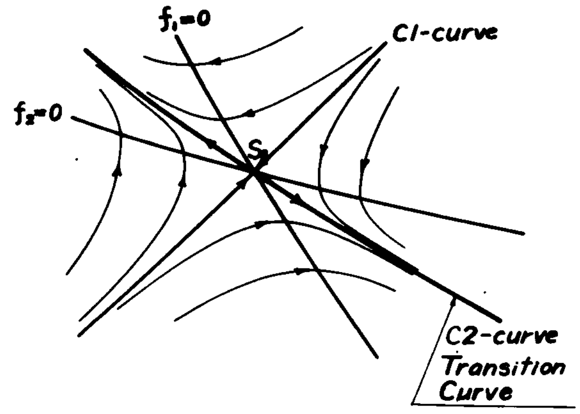

Fig. 2-13 (3) Transitional behaviours of flows through saddle point in steep slopes

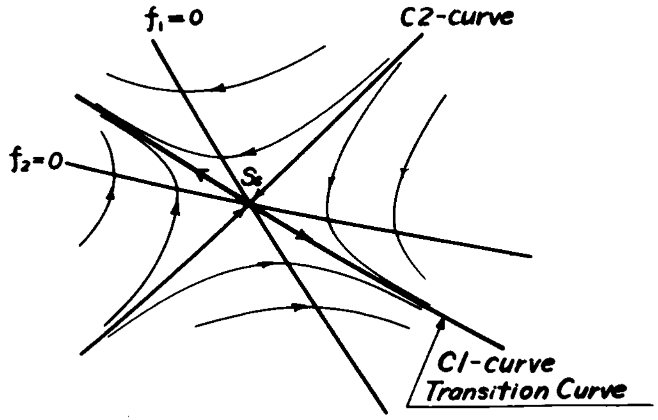

Fig. 2-13 (4) Transitional behaviours of flows through saddle point in mild and adverse slopes

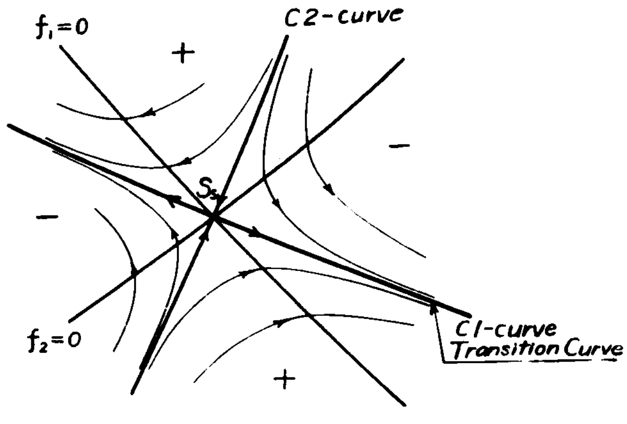

Fig. 2-13 (5) Transitional behaviours of flows through saddle point in mild **and** adverse slopes

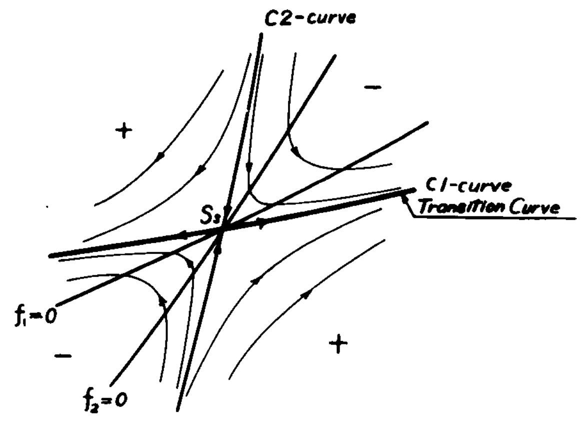

Fig. 2-13 (6) Transitional behaviours of flows through saddle point in mild and adverse slopes

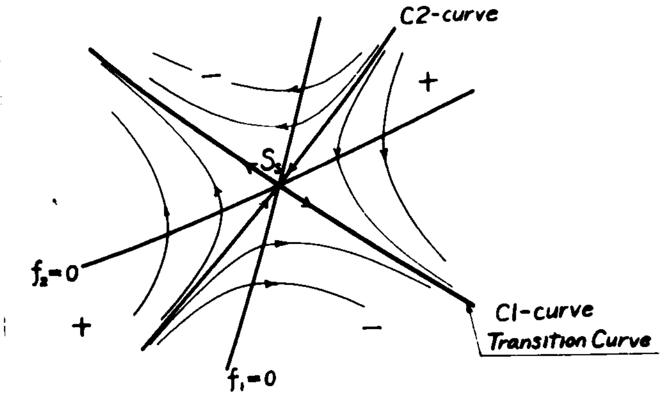

Fig. 2-13 (7) Transitional behaviours of flows through saddle point in adverse slopes

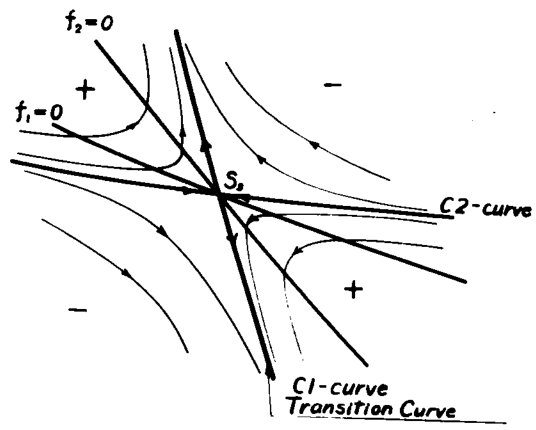

Fig. 2-13 (8) Transitional behaviours of flows through saddle point in adverse slopes

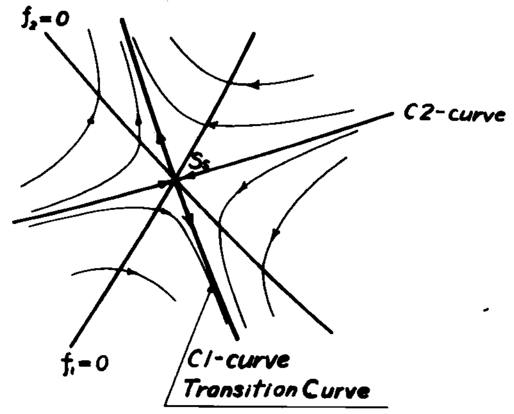

Fig. 2-13 (9) Transitional behaviours of flows through saddle point in adverse slopes

(2) Transitional characteristics by nodal point

For the nodal point, the discriminant of characteristic equation is positive, $(ad - bc)$ is also positive and the flow changes from shooting to tranquil, resulted in that the sign of $(a + d)$ must be equivalent to that of b. In the same manner as did for the saddle point, the transitional characteristics by the nodal point are classified into five types as tabulated and illustrated in Fig. 2-14 (1) - (5). In this case, however, the follwoing attention must be borne in mind.

(i) $a + d > 0$,

$S_2 - a > 0$, if $(a - d) < 0$, and if $(a - d) > 0$, $S_2 - a \gtrless 0$, depending on $bc \gtrless 0$.

(ii) $a + d < 0$,

$S_2 - a < 0$, if $(a - d) > 0$, and on the contrary if $(a - d) < 0$, $S_2 - a \lessgtr 0$ for $bc \gtrless 0$.

Transitional Characteristics of Chézy Flows by Nodal Point

Channel grade	Channel characteristics						Transition curve ($\dot{d}h/dx$)	Figure	Remarks
	a	b	c	d	s_1	s_2			
Steep	+	-	+	-	+	+	+	(1)	
	+	-	-	-	-	+			$(ad - bc) \not> 0$
	-	-	+	-	+	-	+	(2)	
	-	-	+	-	+	-	-	(3)	
	-	-	-	-	-	-	-	(4)	
Mild	+	+	+	+	-	-	-	(4)	
	+	+	-	+	+	-	+	(2)	
	+	+	-	+	+	-	-	(3)	
	-	+	+	+	-	+			$(ad - bc) \not> 0$

	- + - +	+ +	+	(1)	
Adverse	+ + + +	- -	-	(1)	
	+ + + -	+ -			(ad - bc) $\not>$ 0
	+ + - +	+ -	+	(2)	
	+ + - +	+ -	-	(3)	
	+ + - -	- -	-	(5)	
	- + + +	- +			(ad - bc) $\not>$ 0
	- + + -	+ +			b > 0, (a + d) < 0
	- + - +	+ +	+	(1)	
	- + - -	- +			b > 0, (a + d) < 0

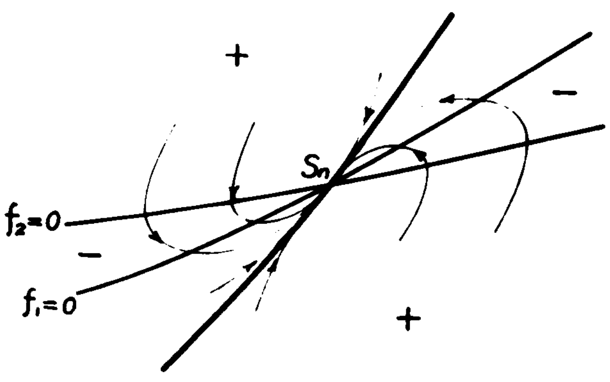

Fig. 2-14 (1) Transitional behaviours of flows by nodal point in steep, mild and adverse channels

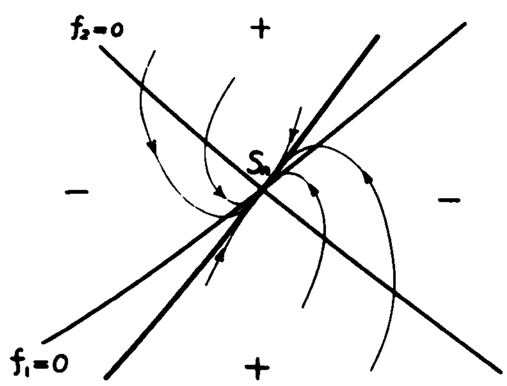

Fig. 2-14 (2) Transitional behaviours of flows by nodal point in steep, mild and adverse channels

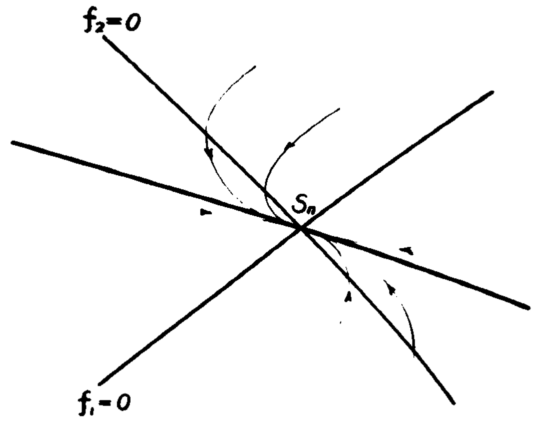

Fig. 2-14 (3) Transitional behaviours of flows by nodal point in steep, mild and adverse channels

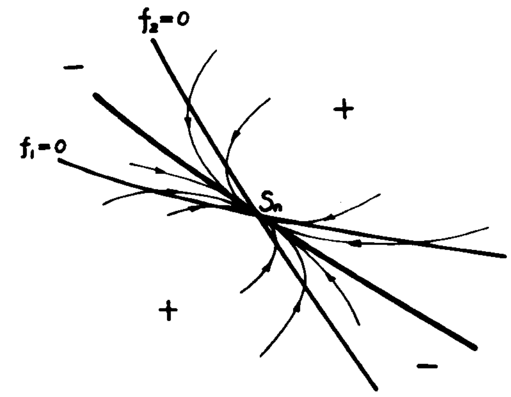

Fig. 2-14 (4) Transitional behaviours of flows by nodal point in steep and mild channels

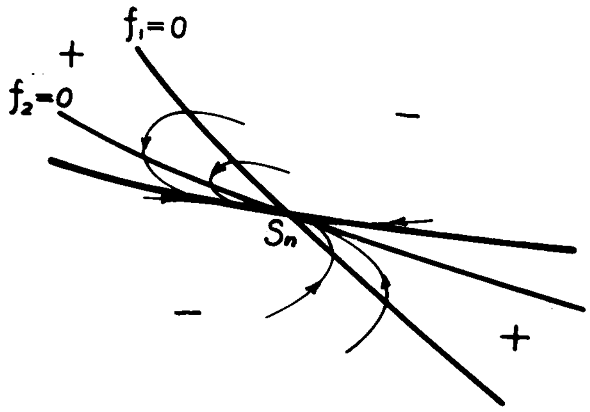

Fig. 2-14 (5) Transitional behaviours of flows by nodal point in adverse channels

(3) Transitional characteristics by focal point

In the same manner, the transitional characteristics through a sudden hydraulic jump produced by the focal point are listed in the following and classified into four types. Figs. 2-15 (1) - (4) describe the surface profiles in the immediate vicinity of the focal point. In figures as well as figures of nodal point, the location of hydraulic jump is not indicated, and at the point where the up- and downstream depths become conjugate, the hydraulic jump can be occured.

Transitional Characteristics of Chézy Flows by Focal Point

Channel grade	Channel characteristics a	b	c	d	s_1	s_2	Figure	Remarks
Steep	+	-	+	-	+	+	(1)	
	+	-	-	-				$(a - d)^2 + 4bc \nless 0$
	-	-	+	-	+	-	(2)	
	-	-	-	-				$(a - d)^2 + 4bc \nless 0$
Mild	+	+	+	+				$(a - d)^2 + 4bc \nless 0$
	+	+	-	+	+	-	(2)	
	-	+	+	+				$(a - d)^2 + 4bc \nless 0$
	-	+	-	+	+	+	(1)	
Adverse	+	+	+	+				$(a - d)^2 + 4bc \nless 0$
	+	+	+	-				$(a - d)^2 + 4bc \nless 0$
	+	+	-	+	+	-	(2)	
	+	+	-	-	-	-	(3)	
	-	+	+	+				$(a - d)^2 + 4bc \nless 0$
	-	+	+	-				$(a - d)^2 + 4bc \nless 0$

-	+	-	+	+	+	(1)
-	+	-	-	-	+	(4)

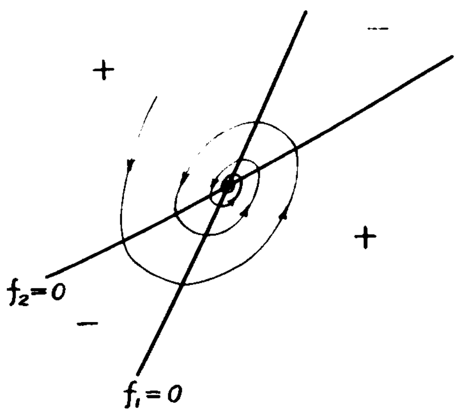

Fig. 2-15 (1) Transitional behaviours of flows by focal point in steep, mild and adverse channels

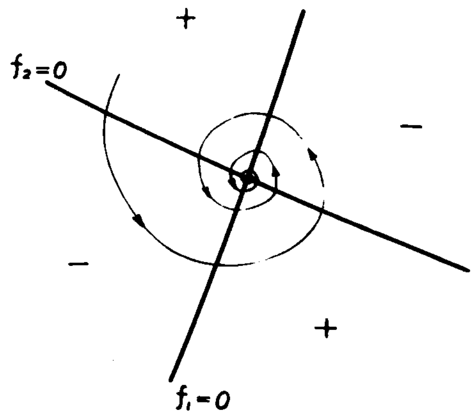

Fig. 2-15 (2) Transitional behaviours of flows by focal point in steep, mild and adverse channels

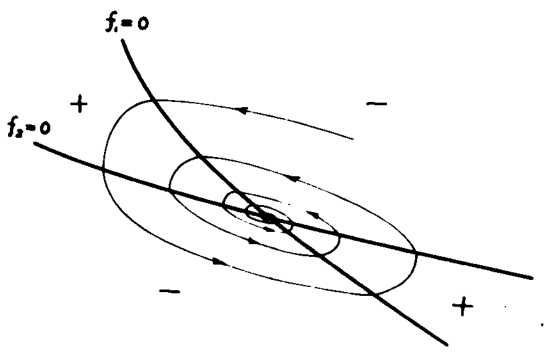

Fig. 2-15 (3) Transitional behaviours of flows by focal point in adverse channels

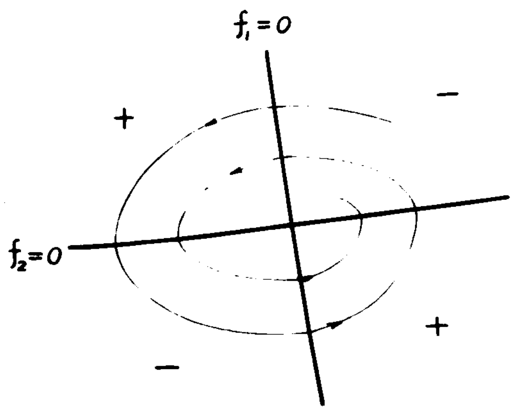

Fig. 2-15 (4) Transitional behaviours of flows by focal point in adverse channels

(c) Some Comments of Transitional Characteristics of Manning Flows

As has been indicated in Part I, the Manning type of boundary resistance is more available for the resistance law of open channel flows than that of Chézy. In alluvial channels, especially, the former becomes extremely useful. In this section, therefore, some comments of transitional characteristics of Manning flows are described.

The Manning roughness n is closely related to the Chézy roughness C through the relationship of $C = R^{1/6}/n$, so that

$$(\partial C/\partial h)_c = (C_c/6R_c)(\partial R/\partial h)_c, \text{ and}$$

$$(\partial C/\partial x)_c = (C_c/6R_c)(\partial R/\partial x)_c - (C_c/n_c)(\partial n/\partial x)_c.$$

The coefficients c and d in the basic homogeneous equation (6) near the singular point, thus, change as follows.

$$c = \cos\theta_c(d\theta/dx)_c + (4Q^2/3C_c{}^2R_c{}^2A_c{}^2)(\partial R/\partial x)_c - (2Q^2/C_c{}^2R_cA_c{}^2n_c)(\partial n/\partial x)_c + (2Q^2/C_c{}^2R_cA_c{}^3)(\partial A/\partial x)_c + \cdots,$$

and

$$d = (4Q^2/3C_c{}^2R_c{}^2A_c{}^2)(\partial R/\partial h)_c + (2Q^2/C_c{}^2R_cA_c{}^3)(\partial A/\partial h)_c + \cdots,$$

whereas a and b are not changed.

If the channel roughness is constant throughout the whole reach under investigation, $(\partial n/\partial x)_c$ and $(\partial C/\partial x)_c$ become zero, and therefore the coefficients of $(\partial R/\partial x)_c$ and $(\partial R/\partial h)_c$ in Manning flows are 4/3 times greater than those in Chézy flows, and the resulting influences in the transitional characteristics are derived by means of the foregoing relationship. Hydraulic characteristics of Manning flows, however, are commonly similar in their essential characters to those of Chézy flows, which will be easily obtained.

2 - 4 - 4 Analysis of Surface Profiles of Water in Open Channels with Relation to Practical Problems in Channel Design

In previous chapters, hydraulic characteristics of gradually varied flows in uniform and non-uniform channels as well as transitional characteristics have been definitely explained. Canals and ditches on irrigation and navigation projects and natural channels on river improvement projects are planned to carry safely required design discharge as specified by the hydraulic study and the economical consideration of construction cost characterized by topography and geology and the future operation and maintenance cost must be equally added. The most favourable design and layout will be then the most economical combination satisfied by the basic requirements. Restricting the problem only to the hydraulic design, the hydraulic procedure for design of open channels is evidently one of water surface determination. The required height of side wall must be determined by adding the necessary freeboard to the carefully estimated water depth. The cross section and the channel grade of the final design determined by a comparison of the combined excavation and other costs for several tentative layouts must ensure the safety-carriage of water and the prevention of bed scour by the excessive force of flowing water.

The flow is usually non-uniform and accelerative or decelerative by the local influence even in uniform channels, and the uniform condition in flow is rarely obtained. Furthermore, in natural channels where the channel geometry and channel grade as well as roughness are continuously changing from point to point, the continuous variation in flow stage and velocity is produced and thus the resulting surface profile is a combined feature of transitional characteristics in flow.

The determination of the water surface curve is usually based on the energy conservation theorem of Bernoulli as indicated in Eqs.(1), (2) and (3) in 1-2-2. The description of basic equation expressed by the momentum approach, however, becomes definite owing to more complex indication by the energy approach in the gradually varied flow. The most important relationship should be established before the calculation of surface profiles is to trace both curves of normal and critical depths. If the channel is of uniform form and the bottom grade and the channel roughness are constant, both curves can be explicitely determined and they classify the channel mild, steep, critical, adverse and horizontal, depending on their geometrical location to each other. In the subcritical part, the flow becomes a downstream control, and therefore the calculation of surface profiles must be proceeded from the downstream end while in the shooting part, the calculation is proceeded from the upstream end, as the flow can not be influenced by the downstream disturbances as has been described. In such channels, as the singular points are not resulted in the basic gradually varied equation, so many fruitful tabled and charts developed by Bakhmeteff, N. Mononobe[7)], Chow and others are useful means for the evaluation of surface profiles. Various methods of numerical analysis and graphical procedures as described in the foregoing chapter are also available for this purpose. If the channel grade and the channel section change at some particular points, surface profiles for the flow of given discharge rate are also determined by computing separately and successively the change in surface elevation in each of a large number of small reaches of the total part of the profile. These reaches must be short enough to reduce, to a permissible magnitude, the errors in approximation of real surface slope through the reach by the average of the surface slopes at each end, or the slope corresponding to the average of the hydraulic characteristics in the reach as well as in considering the irregularities of the channel itself.

Herewith, the hydraulic significance of transitional characteristics resulted from the mathematical properties of singular points in the basic equation will be arised. A saddle point changes the flow from tranquil to shooting, the surface profile passing through the point can be uniquely evaluated, and all other surface profiles, speaking in hydraulic engineering, regulated by other control structures, and mathematically, located in each quardrant separated by two possible singular solutions, can not pass to the other zone without changing its regime by hydraulic jump produced by the singular point. Nodal and focal points may occur the hydraulic jump in the flow. Therefore, the estimation of surface profiles near the singular point must be carefully made, bearing in mind the transitional characteristics of flows.

In this section, the analysis of surface profiles in natural channels and artificial canals and its hydraulic significance to the practical problems in channel design will be concerned, with some examples of surface profiles of gradually varied flows, as the hydraulic design in function is mainly based on the determination of true feature of surface profiles.

Fig. 2-16 illustrates an example of a possible family of surface profiles obtained by the saddle and nodal points. If the flow is assumed of Chézy type, the channel characteristics are that a, b, c and d are positive in mild and adverse channels and all values are negative in steep channels, as seen from the table in the previous section. More definitely in steep channels, it is indicated the channel is convergent and furthermore rapidly decreases its width, and the bottom grade is near critical and gradually decreases in the down-stream direction. In mild channels, the condition of channel characteristics is that the channel geometry is divergent and commonly of gradual expansion, and the grade usually decreases downstream. In adverse channels, the description for channel geometry satisfied by the condition indicated is that the

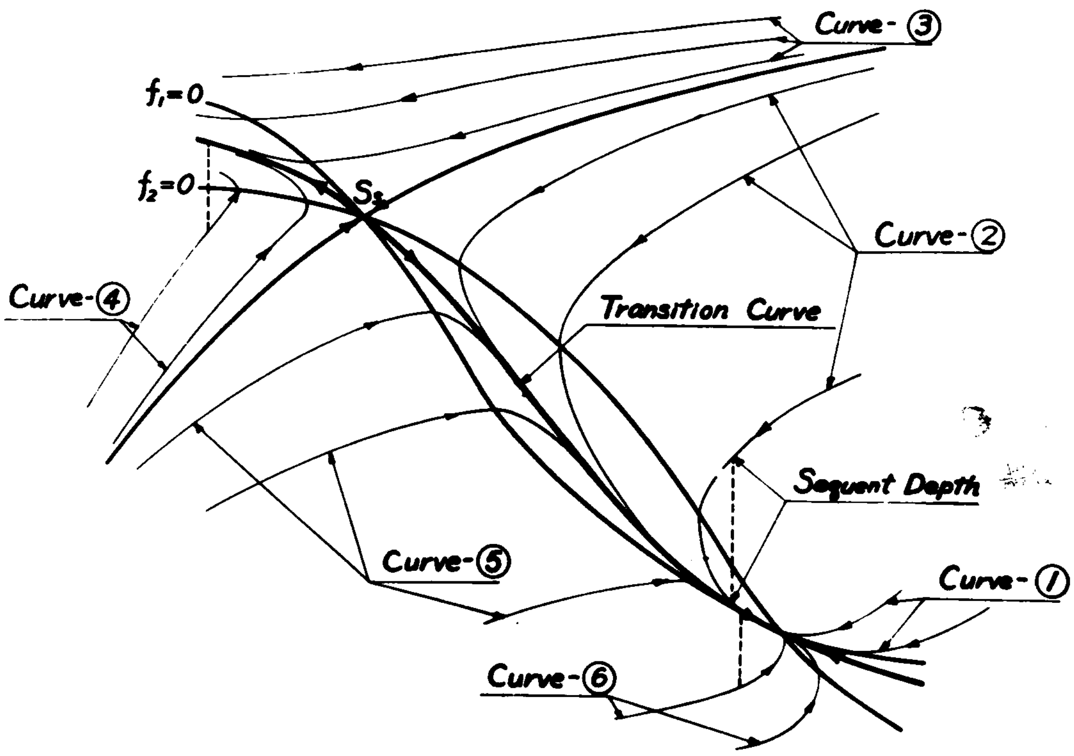

Fig. 2-16 Example of surface profiles of Chézy flows in channel transitions and controls

channel is divergent with gradual expansion similar to the case of mild channels and the bed grade is increasing or gradually decreasing. In such channels of various types in channel geometry and grade, the gradually varied flow will produce the occurrence of saddle and nodal points. If the flow is not regulated by other control structure like dam and gate, the surface profile is represented by singular solutions defined as the transition curve. In the up-and down-stream reaches from the saddle point, the calculation must be started from the point, as shown by the arrow direction, while in the downstream reach from the nodal point, it is proceeded by the downstream end as controls. Evidently, the flow is tranquil in the upstream reach from the saddle point and the downstream reach from the nodal point, and shooting in the reach between two singular points. It is apparently understood that the C1-curve represents a transition curve at the saddle point for mild and adverse channels while the C2-curve a transition curve in the steep channel. By the existence of saddle pôint, the flow characteristics are definitely calculated for the given rate of discharge, so that the channel becomes a control structure. In this case, furthermore, a quite interesting phenomenon will be appeared by the nodal point. The transition from shooting to tranquil is usually occured by the abrupt change in water stage known as the hydraulic jump in uniform and non-uniform channels. As seen apparently in the figure, the possibility of smooth transition from shooting to tranquil without hydraulic jump will be arised (curve 1). It is theoretically verified from a simple consideration to the basic equation of gradually varied flows that the downstream regulated stage by the control structure is less than the curve of $(d^2h/dx^2) = 0$ makes such a transition possible. Experimental verification of this phenomenon is very difficult owing to its small value in regulated stage required for a smooth transition from shooting to tranquil. Further precise measurement must be needed for the verification of this interesting behaviour.

The analysis of surface profiles in reservoir characterized by a dam as a control structure will be next concerned. Most reservoir projects as well as irrigation projects are planned in the mountaneous districts, where the river grade is rather steep and the channel is continuously changing in its cross section, grade and so on. In the transition reach from a reservior to original channel, therefore, the occurrence of intersection between two curves of normal flow and critical depth and the existence of singular points will be possible. Let consider, in Fig. 2-16, that the water elevation regulated by the control structure downstream is less than the stage of C2-curve in mild and divergent channels and of C1-curve in steep and convergent channels. Apparently, the flow regulated by the downstream control is tranquil, so that the calculation of surface profiles is proceeded from the downstream end and the initial value for calculation is the reservoir stage. The surface profile is in a domain enclosed by C1- and C2-curves, and can not pass in itself to the other three domains through singular solutions. In this case, curves as the mathematical solution of basic equation are represented by curves ②, and the downstream influence can not also be transmitted upstream from the saddle point. The saddle point, therefore is a channel control. The procedure of computation is divided into two parts: The saddle point is a control and thus a starting point, from which the calculation must be proceeded up- and downstream directions. Another branch of surface profiles is computed from the downstream end. The combination of two surface profiles is made by the hydraulic jump at a point where the up- and downstream depths become sequent by the momentum conservation law as explained in the previous chapter (The dotted line indicates the location schematically.), and the final surface profile is obtained. The downstream influence evidently ends at the point where the hydraulic jump takes place, and it is also the upstream end of back water zone in reservoir. On the other hand, if the regulated elevation is deeper than the elevation of C2-curve, the saddle point, which was a channel control in the former case, is not a control but a transition, the downstream influence can travel upstream from the saddle point as seen in curve ③ and the back water zone stretches further upstream.

As illustrated in the example, curve ③ is substantially different in its mathematical and physical characters from curve ② . This conclusion is of basic significance for the evaluation of back water zone in reservoirs. In actual channels and reservoirs where the channel characteristics in geometry, grade and roughness are continuously changing, many singular points will be observed at the transition reach, so that much errors will be also involved for the computation procedure, if the theory of transitional characteristics in natural channels is not carefully applied to the problem for the establishment of pertinent design in hydraulic engineering projects, and in fact, such considerations have not been included in many reservoir projects. For this purpose, the following items must be completely analyzed before the determination of surface profiles.

(1) Trace the normal flow and critical depth curves with sufficient accuracies for engineering purposes.

(2) Locations of singular points must be carefully determined. As it is usually insufficient to determine the location by given data, so supplementary observations are needed near the singular point.

(3) Classify the singular points, following by the foregoing description.

(4) Compute the surface profiles by the basic principles of open channel flows.

The usual procedure of computation in natural channels is the step by step method, and errors are cumulatively involved when the calculation is proceeded. Much attention will be especially called near the singular point. Some evidences will be illustrated in the later part of this study.

The analysis of surface profiles of underflows regulated by a gate as a control structure and associated transitional characteristics will be explained. The surface profiles of downstream underflows are classified into curves ④, ⑤ and ⑥ by the location of gate and the gate-opening. If the curve (4) is the surface profile under investigation, the saddle point becomes a channel control. The hydraulic jump occurs at a point where the curve (4) and the transition curve traced from the saddle point are conjugate in a reach between the gate and the saddle point. After passing through the saddle point, the surface profile is prepared by the transition curve, and therefore the flow initially shooting changes to tranquil by the hydraulic jump, again to shooting in the downstream from the saddle point, and finally becomes tranquil. When the gate is gradually closed, the surface profile from the gate is represented by the curve (5). In this case, the saddle point can not become a channel control, and the flow is shooting before the elevation of underflow from the gate and the water stage traced from the downstream end are sequent together. The flow, therefore, changes from shooting to tranquil through the hydraulic jump occured in a reach between the saddle and nodal points. In this case, if the downstream stage is deep and its influence can be transmitted upstream from the saddle point, the flow is known as drowned or submerged as indicated in uniform channels. By small opening of gate, the curve (6) represents the surface profiles of underflows. The general behaviours of curve (6) are similar to those of curve (5). The magnitude of kinetic energy, however, in the flow is extremely large, so that the hydraulic jump may take place at the downstream reach from the nodal point. The drowned flow will often appear under the condition of high downstream stage. In this case, the underflow which can not be indicated in the basic equation will show a strong action to the bed scour. Careful protection from the scouring by the energy dissipator must be prepared in the structure.

Another example of surface profiles will be seen in Fig. 2-17. The channel characteristics are of same type as in the foregoing case.

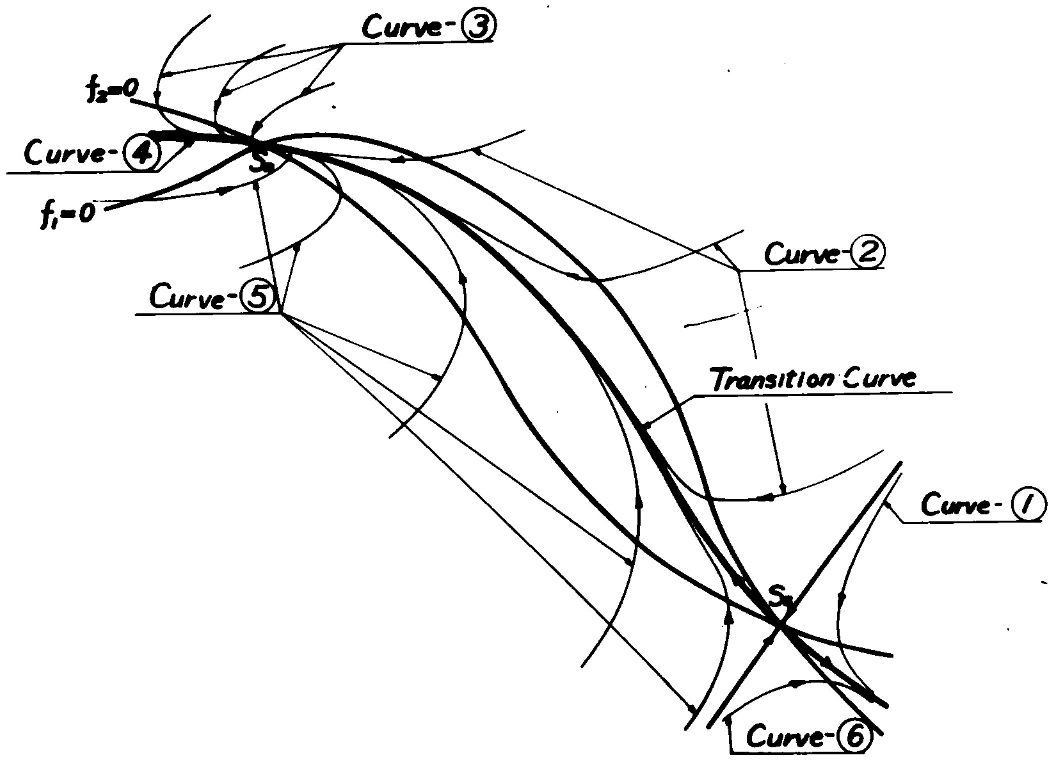

Fig. 2-17 Example of surface profiles of Chézy flows in channel transitions and controls

It is illustrated when the nodal point first appear and is followed by the saddle point. If the flow is approximately considered two dimensional in a channel of constant grade and thus the hydraulic radius is practically equivalent to the flow depth, as treated completely by the author[3)], the former case concerns with the transition flow in mild and divergent channels, while the latter that in steep and convergent channels.

Evidently, the saddle point is a starting point and the nodal point is a terminal point for the calculation procedure. The transition curve indicates the initially shooting flow changes its regime to tranquil through the nodal point and again to tranquil through the saddle point. All curves above the transition curve describe the flow behaviours of surface profiles regulated by the downstream control. If the curve (1), which indicates the surface profiles of small downstream stage, is considered, the hydraulic jump takes place in the downstream reach from the saddle point. The back water influence is terminated by the jump. The curve (2) terminates at the nodal point. In this case, no hydraulic jump occurs in the whole part under investigation, and apparently the back water zone can not be transmitted to the upstream reach from the nodal point because the nodal point is a terminal for computation. When the downstream stage becomes deeper, the influence still stretches from the nodal point and is ended by the hydraulic jump.

Curves of (4), (5), and (6) indicate the flow behaviours of surface profiles regulated by the upstream controls. The curve (4) describes the downstream surface profile controlled by a weir or a dam in steep channels, while other two curves those of underflows from a gate. If the surface profile is expressed by the curves of (4) and (6), no hydraulic jumps occur in the flow. On the other hand, the initially shooting underflow regulated by the medium opening of gate changes its flow regime to tranquil by the hydraulic jump and thereafter the flow behaviour is determined by the downstream control of a saddle point or control structures.

The example in which the focal point is produced will be treated. In this case, as seen in Fig. 2-18, the channel characteristics are given by positive values in a and b and negative ones in c and d. Apparently, the channel is of adverse grade and divergent. Being different from the preceding two cases, the smooth transition curves in the whole reach can never be produced owing to the existence of focal point. Curves of (1) and (2) represent surface profiles traced by the downstream control, and curves expressed by (1) indicate the surface profiles of low regulated elevation in water stage and can

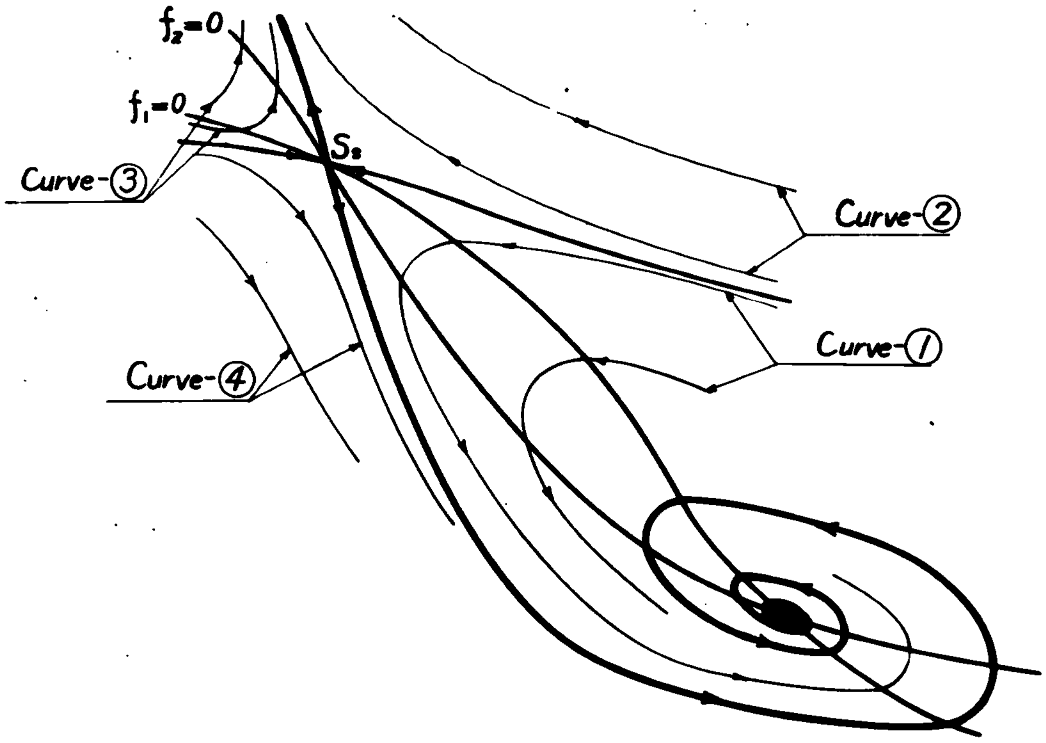

Fig. 2-18 Example of surface profiles of Chézy flows in channel transitions and controls

not travel upstream from the saddle point, while the curve ② computed from the downstream control with high stage transmits its disturbance and the resulting back water upstream. In this case, the focal point can play no role in the flow behaviour. Curves ③ and ④ describe the surface profiles of flows controlled by the upstream structures from the saddle point. The curve ③ is combined with the local transition curve traced from the saddle point through the hydraulic jump and again connected with the downstream branch through the jump after passing through the saddle point. On the other hand, the curve ④ involves only one hydraulic jump in its flow behaviours.

As seen in three examples illustrated schematically, the general features of transitional characteristics of open channel flows and resulting surface profiles in natural or artificial channels changing continuously their channel geometry, grade and roughness are of great complexity. Consequently, the hydraulic significance of transitional characteristics to practical problems in hydraulic engineering will be easily understood. Because, as often explained, the basic requirements of hydraulic design of structures are substantially based on the complete determination of surface profiles with sufficient accuracies satisfied by the engineering purpose.

In the preceding discussion treated with the first order theory of gradually varied flow, the influence of vertical acceleration yields the formation of waves in the flow near critical regime, when the non-uniform flow approaches normal and tranquil downstream, so that the smooth transition flow from shooting to tranquil through the nodal point will be possibly indicated by a successive series of waves. The transition from undular to normal jump, therefore, will be generally expected for the open channel flow in such channel transitions and controls.

---

1) Massé, P., Ressaut et ligne d'eau dans les cours d'eau à pente variable, Rev. gen. Hydraulique, Nos. 19-20, Jan. - Apr. 1938.

2) Escoffier, F.F., Transition Profiles in Non-Uniform Channels, Jour. Hydraulics Division, Proc. ASCE, HY 3, Jun. 1956.

3) Iwasa, Y., Theoretical Study of Hydraulic Behaviours of Boundary Characteristics to Channel Transitions and Controls in Divergent or Convergent Channels, Trans. JSCE, No. 59, Separate 3-1, Nov. 1958 (in Japanese).

4) Iwasa, Y., Hydraulic Significance of Transitional Behaviours of Flows in Channel Transitions and Controls, Memoirs, Fac. Eng., Kyoto University, Vol. 20, No. 4, Oct. 1958.

5) Jaeger, C., Engineering Fluid Mechanics, Blackie, London, 1956.

6) Homma, M., Hydraulics (Fluid Mechanics for Hydraulic Engineers), Maruzen, Tokyo, 1955 (in Japanese).

7) Mononobe, N., Back-Water and Drop-Down Curves for Uniform Channels, Trans. ASCE, Vol. 103, 1938.

III. HYDRAULIC PERFORMANCES OF CONTROL STRUCTURES IN CONNECTION WITH HYDRAULICS OF CONTROL SECTION

1. General Characteristics of Hydraulic Diversity of Control Structures in Functional Service

As has been indicated in the previous parts, the surface profiles of open channel flows are considerably influenced by continuous changes in crocs section, channel grade and boundary resistance, and such influences may be in common called as transitional behaviours by channel transitions. Therefore, a channel transition is defined as a local change in channel geometry, grade and boundary resistance, which produces a variation in flow one uniform state to another, following by the definition of A.T. Ippen[1)]. The resulting surface profile, which is changed from point to point in channel transitions, is described as a combined feature of transitional behaviours.

A channel control is defined as the class of transition for which the elevation of the water surface and thus all other hydraulic characteristics of open channel flows can be uniquely predicted at a particular cross section for the given rate of discharge, and the section predicted is also called as the control section. Weirs, sluices, spillways and some kinds of flumes may be cited as examples in which the surface elevation in the vicinity of the structure can be predicted. The hydraulic significance of such a class of transitions has been already explained. Speaking in mathematical hydraulics, three kinds of singular point may possibly become a channel control. Apparently, the saddle point, which induces the flow from tranquil to shooting with a smooth change in flow pattern, becomes a control. The nodal point, which is a transition from shooting to tranquil, induces generally the hydraulic jump near the point, and in some particular cases a smooth transition through the nodal point becomes possible. When the hydraulic jump is occured, the hydraulic characteristics can not uniquely be calculated in common and thus it is not a control but only a transition, whereas in the latter case, the nodal point also can be a control. On the other hand, the focal point, by which the hydraulic jump is always resulted, can not become a control.

A control structure is defined as the class of structure to carry water, in which the flow has the control section characterized by the unique determination in hydraulic behaviours, and therefore the control structure is the stage-discharge regulating device in hydraulic engineering. Typical examples of control structures are sharp and round crested weirs and spillway crests, which are of form of a weir in rapidly varied flows and broad crested weirs and Parshall flumes in gradually varied flows. Gates are also examples of control structures owing to regulating action of water elevation. Limiting the problem subjected to the control behaviours of hydraulic structures for open channel flows, the detailed function of control structures to the flow is described separately in the following.

Weirs and flumes in open channels, especially in irrigation canals are usually used for the following different objectives[2].

(1) For the proper distribution of the water carried by a main canal among the branch canals depending on it. In this problem, the basic equation of flows are changed by the side in- and outflows and studied by de Marchi, Iwagaki, Homma and others. With the application of mathematical procedures described in the foregoing chapter dealed with the transitional characteristics by means of the geometric theory of basic differential equation, the proper distribution of the water is sufficiently estimated for the irrigation works.

(2) For reducing the hydraulic slope in a canal, should the natural slope of the ground surface be greater than the maximum hydraulic slope permissible from the consideration regarding prevention of undesirable erosion.

(3) For reducing the head on an existing structure, which, for some reasons or other, may have proved too weak for its purpose.

(4) As a sand-screen to exclude heavy sediment from the canal.

(5) For measuring the discharge.

Of special significance in theoretical hydraulics of the hydraulic characteristics of control section is the flow measurement of open channel flows by the use of weirs and flumes. If the channel transition usually consisted of contraction and expansion in channel geometry and changes in channel grade is used for the purpose of discharge measurement, the flow condition is not changed in channels, so that the water-level measurements are required for the evaluation of discharge rate at two points in a channel. On the other hand, when the channel transition is used as the control structure, the flow regime is substantially changed from tranquil to shooting, or in some particular cases, from shooting to tranquil without the formation of hydraulic jump of violent turbulence. Furthermore, the flow characteristics at the control section can be uniquely calculated for the given rate of discharge. As it ensures the independence of the approach flow on downstream conditions, so the discharge rate can be evaluated by the single water-level measurement, as done in weirs, and this principle gives the most important requirement for the hydraulic design of control structures.

Functional diversities in the hydraulics of control performance make various types of control structures. Parshall flumes, which are essentially used as low head device for discharge measurement in irrigation works, are characterized by the gradually varied flow assumed by the prevailing hydrostatic pressure in flow. On the other hand, weirs and overflow spillways are influenced in their hydraulic behaviours by the rapidly varied flow of curvilinear motion, which is considered as a counterpart to the theory of gradually varied flow mainly developed by the scientists in the 19 th century. For low head devices in irrigation works at flat area, the flume is commonly available, whereas the weir is used for laboratory equipments and high head devices, as the tremendous amount of experimental data has been published for the hydraulics of weir flows.

The hydraulic performance of round crested weirs as the control structure is first concerned by means of the theory of control section described in the foregoing, with the systematic experimental data obtained at the Hydraulics Laboratory, Kyoto University. In the next chapter, that of overflow spillway is treated. The problem has been commonly concerned by the scale model tests for various purposes relating to the valley development projects. The theory described will ensure the possibility to make the general characters of flow behaviours over a control structure of spillway clear. The hydraulics of the head-discharge relationship of flows over a sharp crested weir and a sill is followed. It is a subject of oldest hydraulic researches since the 18 th century. Nevertheless, the hydraulic performance of this type of control structure has not been clarified owing to its greatest complexity in flow behaviours, and the basic flow pattern is not subject to a complete mathematical description. In this chapter, the head-discharge principle by a sharp crested weir will be considered by means of the theory of transitional characteristics of curved flows, as a tentative attempt to obtain the universal law in discharge equation. The last chapter will treat with the function and performance of flumes as typical examples of control structures for gradually varied flows, as shown in the above statement. Within the validity of basic assumption implied in the gradually varied flow, the hydraulic performance of flumes will be revealed in an implicite form.

The general scheme of design problems in control structures as a part of the whole hydraulic project is still to try successively varying combination of elements in hydraulic and structural functions until the most available combination in economy is found. In some cases, as seen in the spillway design, however, the structural design of control structures will become a primary element. Much attention must be called that the structure is useless or inversely plays a wrong device to assure the safety-carriage of water-release, if the complete consideration to the structure in hydraulic performances is not made.

---

1) Ippen, A.T., Channel Transitions and Controls, Engineering Hydraulics, edited by H. Rouse, John Wiley, New York, 1950.

2) Leliavsky, S., Irrigation and Hydraulic Design, Vol. 2, Irrigation Works, Chapmann and Hall, London, 1957.

2. Hydraulics of Round Crested Weir

3 - 2 - 1 Basic Consideration

The first stage for the analysis of hydraulic performance of control structures to the open channel flow will concern the hydraulics of round crested weir. For many years, with the advance of history of hydraulic research, the definite comprehension of a relationship between discharge and head by various forms of weirs have been widely sought. Among a large number of types of weirs, the sharp crested weir is most popular. Nevertheless, the complete analytical conclusion has not yet been obtained, as it is not subject to exact mathematical description for the physics of weir flows. On the other hand, the hydraulics of flow behaviours over a round crested weir will be expressible in mathematical form, as seen in the next, so that the desirable relationship of discharge to head is also established, and furthermore, the flow characteristics will furnish the verification of critical depth theory in terms of maximum discharge and minimum energy developed since Bélanger and Bresse over a curved boundary.

Adding to the especial significance in the basic hydraulics of fluid flows, this type of weir is also important as a safety carriage device for released water flow in some particular types of weirs. For examples, the new Nango Weir, which is now under reconstruction at the River Seta, Shiga Prefecture, Japan, is equiped with double lift gates. When the quick discharge for flood emergency is needed, both of separated gates are operated, whereas in ordinary time, the discharge will be released from the upper gate in a form of overflow. Under such situation, the upper gate is provided by a curved vane for protecting the lower member from the excessive impulsive force of stream flow as well as for obtaining the more efficiency in dis-

charge relationship.
The round crested weir will be defined as a type of weir with curved solid boundaries which can guide the flow. As a special type of this weir, the weir with a circular boundary of constant radius is simplest in hydraulic design and formulation of flow behaviours. If the curved section of weir is connected by the long discharge carrier, the hydraulic behaviours of flow will be in the category of spillway hydraulics, and in this case, the curved section is physically called as a control structure, as it will surely become a channel control for all rates of released discharge from reservoir. The hydraulics of round crested weir, therefore, is considered as a basis of hydraulic characteristics of spillway crests.

The basic requirements for selection of shapes in curved boundary section are determined by the efficiency of weir in released discharge and the future maintenance cost due to the damage in materials of boundary. Apparently, the former need is closely related to the flow characteristics which are main subject of the present study and the latter to the pressure distribution along the boundary itself. As in design problems of overflow spillways and other conveyance structures, the usual nappe shape is commonly based on the so-called Creager curve which has been obtained by Creager through the systematic experiments of free nappe flows. Nevertheless, the curve of Creager is not expressed in a mathematical form, so that the analytical treatment for above described purposes also can not be made. For many years, therefore, the relationship between discharge and head has been confirmed by the experimental investigation of scale models, many empirical relationships were obtained for particular shapes of weir geometry, and the unified analysis and general treatment are not established.

The flow over a round crested weir is evidently curvilinear flow, and the regime of flow changes from tranquil to shooting . The change of flow regime is rather rapid, so that the analysis of flow behaviours will be classified as the rapidly varied flow and the curved boundary flow, though Jaeger classified the flow was a typical example in the category of gradually varied flow.

The analytical study in the hydraulics of curved flows over a round crested weir initiated by Jaeger[1)] in 1938. With the use of velocity distribution in curved potential flow, the head-discharge relationship as a parametric expression of curvature of boundary was obtained. This relationship is confirmed by experimental data obtained at universities of Darmstadt, Lausanne, Münich and others. The basic concept of treatment by Jaeger is essentially derived by the critical depth theory of Bélanger and Böss, as has been described in 1-2-5. Without a rigorous mathematical proof, he anticipated the flow became critical at a section in the boundary. Of course, the control section will be observed in the flow, but the proof that the round crested weir serves as a control structure for all rates of released discharge is particularly necessary.

In the present discussion of analytical treatment in hydraulics of round crested weir, the hydraulic characteristics of flows over a round crested weir are first concerned. Theoretical analysis starts from the potential flow theory as the engineering approximation. This approximation in the basic physics of flow as well as the analytical procedure will be confirmed by the systematic experiments conducted at the Hydraulics Laboratory, Kyoto University. It is also verified that the weir serves as a control device for the flow and therefore the head-discharge relationship is uniquely determined, if the channel geometry in shapes of boundary is given. The detailed discussion will be followed in the next section. The last section of this chapter, the experimental verification of the theoretical analysis of flows over a circular weir will be described to give hydraulic informations when the hydraulic design of this type of weir is planned.

The problem is also applied to the flow for siphons. When designing the inlet and throat of a siphon, C. Mallet and J. Gaussens[2)] used the same approach and obtained the discharge relationship.

3 - 2 - 2 Hydraulic Characteristics of Flows over Round Crested Weir

(a) Basic Characteristics of Flows over Round Crested Weir

Being different from the free flows over a sharp crested weir, the flow over a round crested weir will be guided by the solid boundary to some extent, As the length of boundary, however, is generally short, so the ignorance of shear influence will be possible as the engineering approximation. Irrotationality of fluid flows becomes available for the analysis of hydraulic characteristics of round crested weir. This assumption will be verified by a large number of experimental data of Kyoto University, as briefly illustrated in 1-1-3. The analysis starts from this basic assumption on irrotationality of flow.

Taking the x-axis along the curved boundary and the y-axis normal from the boundary, the irrotational water flow under the field of gravitation is expressed by the following equations of motion, continuity and irrotationality, as seen in literatures of fluid mechanics[3)].

$$uR(\partial u/\partial x) + v(R + y)(\partial u/\partial y) + uv = g\sin\theta(R + y) - R(1/\rho)(\partial p/\partial x), \tag{1}$$

$$uR(\partial v/\partial x) + v(R + y)(\partial v/\partial y) - u^2 = -g\cos\theta(R + y) - (R + y)(1/\rho)(\partial p/\partial y), \tag{2}$$

$$(\partial u/\partial x) + v/R + (1 + y/R)(\partial v/\partial y) = 0, \tag{3}$$

and

$$(\partial v/\partial x) - u/R - (1 + y/R)(\partial u/\partial y) = 0. \tag{4}$$

In above equations, the complete solution under given conditions can not be obtained, and therefore, some approximate treatments which will satisfy the engineering purpose, must be made. The velocity component of v in the y-direction is assumed of small order compared with u, so that the first approximations will become as follows.

$$u(\partial u/\partial x) = (1 + y/R)g\sin\theta - (1/\rho)(\partial p/\partial x), \tag{5}$$

$$-(1/\rho)(\partial p/\partial y) = g\cos\theta - u^2/(R + y), \tag{6}$$

$$(\partial u/\partial x) + v/R + (1 + y/R)(\partial v/\partial y) = 0, \tag{7}$$

and

$$u/R + (1 + y/R)(\partial u/\partial y) = 0. \tag{8}$$

The velocity distribution of flow, u, over a curved boundary is approximately determined, as seen in Eq.(27) in 1-1-2, by

$$u = \{q/\log(1 + h/R)\}\cdot(R + y)^{-1}, \tag{9}$$

and it indicates that the product of u and $(R + y)$ is constant as an approximation for velocity profiles. Jaeger[4] assumed the velocity distribution was in a form of

$$(u/u_b) = R^{1/m}/(R + my)^{1/m}, \tag{10}$$

in which m is a constant determined by the velocity distribution. It is seen from the basic relationship of motion, the insertion of $m = 1$ into Eq.(10) yields the velocity profile expressed by Eq.(9). When making the analysis of discharge characteristics of flows over a round crested weir, Jaeger assumed m was nearly equal to 2, based on the experimental data of Fawer, whereas his experimental results indicated $m = 2.2$ in average.

By a large number of experiments on velocity profiles of curved flows conducted at the Hydraulics Laboratory, Kyoto University, m will be assumed unity as an engineering approximation, and consequently the constancy of product of u and $(R + y)$ is applied to the analysis. This indication also was proved by $u(R + y)$ for all runs of experiments. Table 3-1 indicates the products of $u(R + y)$ for various runs of experiments. Values in the immediate vicinity of solid boundary are less than those in upper zones of flow. Nevertheless, it will be practically assumed that the constancy law is valid, as the secondary influences due to the surface resistance of walls and the negligence of higher terms depending on the vertical velocity are small compared with the primary fluid motion. Eq.(7) of continuity gives the velocity distribution of v, if the second approximation is required.

The equation of motion in the y-axis indicates the pressure distribution in the curved flow, and with the use of Eq.(9), it becomes

$$(p/\rho g) = (h - y)\cos\theta + (u_b{}^2R^2/2g)\{(R + h)^{-2} - (R + y)^{-2}\},\qquad(11)$$

as has been indicated. The measurements of pressure distribution in the curved flow were also made at the Hydraulics Laboratory, Kyoto University, and the conclusion describes the measured pressure is larger than that of Eq.(11) near the solid boundary, while the opposite behaviour is seen in the upper zone of flow near the free surface. The difference will be caused by the assumption of irrotationality of flow as well as the accuracy of measurement. In approximations, however, Eq.(11) will be available for the analysis of hydraulic behaviours of flows over a round crested weir.

The one dimensional equation of motion, which is the available mean for hydraulic research of open channel flows, is derived through the direct integration of Eq.(1) or (5). The better way for the hy-

Table 3-1 Velocity Profiles of Flows over Round Crested Weir

(1) Q = 9.40 1/sec.

Distance	X = 15.0 cm		X = 20.0 cm		X = 25.0 cm	
y cm	$u^{cm/sec}$	u(R+y)	$u^{cm/sec}$	u(R+y)	$u^{cm/sec}$	u(R+y)
0.12	133.8	2023	102.5	1550	73.7	1114
0.32	132.7	2033	101.8	1560	74.5	1141
0.52	131.7	2044	101.3	1572	73.5	1141
0.72	131.0	2059	99.9	1570	73.0	1147
0.92	130.2	2073	99.6	1586	72.2	1147
1.12	129.2	2083	99.1	1597	71.3	1149
1.32	128.2	2092	98.3	1604	-	-
1.52	127.1	2100	97.1	1604	-	-
1.62	-	-	-	-	70.3	1168
1.72	125.2	2093	96.8	1618	-	-
1.92	124.2	2101	95.7	1619	-	-
2.12	123.9	2121	94.8	1623	69.3	1186
2.32	121.2	2099	94.1	1630	-	-
2.52	-	-	93.2	1633	-	-
2.62	-	-	-	-	66.8	1177
2.72	-	-	92.4	1637	-	-
2.92	-	-	92.1	1650		
3.12	-	-	-	-	65.7	1190
3.62	-	-	-	-	65.0	1210
3.92	-	-	-	-	63.4	1199

X: distance along the solid boundary from downstrean edge. The diameter of weir is 30 cm, and X at crest is 23.56 cm.

(2) Q = 15.08 l/sec

Distance	X = 15.0 cm		X = 17.5 cm		X = 20.0 cm	
y cm	$u^{cm/sec}$	u(R+y)	$u^{cm/sec}$	u(R+y)	$u^{cm/sec}$	u(R+y)
0.12	145.2	2195	130.6	1975	114.2	1727
0.32	143.1	2195	129.1	1978	113.7	1742
0.52	141.4	2196	129.5	2010	112.0	1739
0.72	140.7	2215	128.3	2020	111.1	1748
0.92	140.0	2230	127.2	2030	110.2	1755
1.12	139.3	2245	127.5	2055	109.8	1770
1.32	138.9	2268	124.8	2038	108.9	1778
1.52	137.9	2280	125.2	2070	108.4	1791
1.72	137.5	2300	125.2	2093	108.0	1808
1.92	136.8	2315	124.8	2110	107.5	1820
2.12	136.8	2343	124.4	2130	107.1	1834
2.32	136.1	2359	123.2	2130	107.1	1855
2.52	136.1	2383	122.4	2145	106.2	1860
2.72	134.3	2380	122.4	2170	106.2	1882
2.92	132.4	2375	121.6	2180	106.2	1903
3.12	130.6	2368	119.6	2168	105.2	1908
3.32	-	-	120.0	2198	104.3	1908
3.52	-	-	118.4	2195	104.3	1912
3.72	-	-	116.7	2185	-	-
3.92	-	-	-	-	102.9	1948
4.32	-	-	-	-	102.4	1980

draulic analysis of open channel flows is to use the one dimensional procedure of approach as seen in 1-1-4. The result, therefore, becomes for the energy approach,

$$H_o = h\cos\theta + (u_s^2/2g) - z, \qquad (12)$$

when the origin of coordinate system is selected at the crest of **weir**, through which the reference line for energy is located, and Eq.(12) is evidently known as the Bernoulli equation for invicid fluid flows. H_o is the total head from the crest and z is the vertical distance from the crest to a point under investigation. With the aid of the Bernoulli equation, the surface and bottom velocities also are expressible as

$$u_s = \sqrt{2g(H_o + z - h\cos\theta)}, \qquad (13)$$

and

$$u_b = (1 + h/R)\sqrt{2g(H_o + z - h\cos\theta)}. \qquad (14)$$

(b) Hydraulic Characteristics of Flows over a Round Crested Weir

This subsection concerns with the hydraulic characteristics of flows over a round crested weir, obtained by the preceding analysis. For this purpose, the surface profile equation will be first deduced from the basic energy equation. Differentiating Eq.(12) with respect to the distance, the resulting equation becomes

$$(dh/dx) = f_1(h, x)/f_2(h, x), \qquad (15)$$

in which

$$f_1(h, x) = \sin\theta\{1 + h(d\theta/dx)\} + (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}$$
$$(dR/dx)\{\log(1 + h/R) - (h/R)\},$$

and

$$f_2(h, x) = \cos\theta - (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}\{1 + \log(1$$
$$+ h/R)\},$$

as $(dz/dx) = \sin\theta$.

The mathematical properties of Eq.(15) describe the hydraulic characteristics of flows over a round crested weir, if a particular value of discharge is given for the definite weir geometry. The round crested weir designed for the discharge measurement must be a channel control in the whole range of discharge requested by the design purpose, so that Eq.(15) involves the singular point classified as a saddle point by the theorem of simultaneous occurrence of maximum discharge and minimum energy as described in the foregoing part. If the saddle point is not involved, the weir will be classified as a channel transition in the hydraulics of open channel flows and the relationship between head and discharge can not be uniquely determined by the theoretical analysis, and consequently the model or prototype experiments for a particular shape of round crested weir are needed for the calibration of discharge characteristics. The statement also gives one of basic hydraulic requirements in channel geometry for the functional design of round crested weir.

At the singular point,

$$\sin\theta_c\{1 + h_c(d\theta/dx)_c\} + (q^2/g)(R_c + h_c)^{-3}\{\log(1 + h_c/R_c)\}^{-3} (dR/dx)_c\{\log(1 + h_c/R_c) - (h_c/R_c)\} = 0, \tag{16}$$

and

$$\cos\theta_c = (q^2/g)(R_c + h_c)^{-3}\{\log(1 + h_c/R_c)\}^{-3}\{1 + \log(1 + h_c/R_c)\}. \tag{17}$$

As the simplest but basic case of round crested weir, the circular weir will be concerned. The curvature of boundary is then constant, and consequently the basic equation of (15) for surface profiles becomes

$$(dh/dx) = f_1(h, x)/f_2(h, x), \tag{18}$$

where

$$f_1(h, x) = \sin\theta(1 + h/R),$$

and

$$f_2(h, x) = \cos\theta - (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}\{1 + \log(1 + h/R)\}.$$

Transforming the dependent variable from x to θ, Eq.(18) is

$$(dh/d\theta) = f_1(h, \theta)/f_2(h, \theta), \tag{19}$$

$$f_1(h, \theta) = \sin\theta(R + h),$$

and

$$f_2(h, \theta) = \cos\theta - (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}\{1 + \log(1 + h/R)\}.$$

Evidently, the location of singular point is determined by

$$\sin\theta_c = 0, \tag{20}$$

and

$$\cos\theta_c = (q^2/g)(R + h_c)^{-3}\{\log(1 + h_c/R)\}^{-3}\{1 + \log(1 + h_c/R)\}. \tag{21}$$

This indication describes the singular point is situated at the weir crest and the head-discharge relationship is uniquely determined by Eq.(21), if it is verified that the singular point is classified as a saddle point. With the use of Taylor's theorem in the immediate vicinity of singular point, Eq.(19) can be approximately expressed by the following relation of

$$(dh'/d\theta') = (c/b)(\theta'/h'), \tag{22}$$

in which h' and θ' are new variables transformed by $h = h_c + h'$ and $\theta = 0 + \theta'$, and b and c are, respectively

$$b = (u_{sc}^4/gq^2)\left[1 + \log(1 + h_c/R) + \{\log(1 + h_c/R)\}^2\right] > 0,$$

$$c = R + h_c > 0.$$

Referring to the description of basic characteristics of transitional behaviours of flows indicated in 1-2-4, coefficients of a and d are zero. Consequently, $bc > 0$, so that the characteristic equation of Eq.(22) has two real roots of opposite signs. The singular point is therefore a saddle point and the flow changes its regime from subcritical to shooting at the weir crest which is a control section, as expected by many hydraulic engineers through their past professional experiences. The circular weir thus serves as the control structure and the theoretical evaluation of head-discharge relationship becomes possible for all discharges.

Another significant hydraulic parameter for the design problem of round crested weir is the bed pressure distribution expressed by Eq.(42) in 1-1-3. The rapid decrease of pressure resulted from the sudden increase of velocity will be a cause of cavity formation on the solid surface and the zero pressure along the boundary indicates the necessary length of curved boundary as a guide of weir flow. The longer boundary yields the unfavourable condition for pressure requirement and thus the structure materials. The detailed consideration necessary for the hydraulic design of this type of weir will be discussed in the last section of this chapter.

3 - 2 - 3 Head-Discharge Relationship of Circular Weir

Eq.(21) is evidently the relationship between discharge and head for the flow over a circular weir and a weir of constant curvature of boundary, and consequently this relationship indicates the Böss theorem for critical depth theory. By means of the theorem proved in 1-2-5 for the relationship of minimum energy and maximum discharge, Eq.(21) also represents the Bélanger theorem for maximum discharge. The direct proof of the simultaneity of maximum discharge and minimum energy is easily made by the differentiation of Eq.(12) with respect to the depth.

The head-discharge relationship of flows over a circular weir is commonly expressed in terms of the well known weir formula by hydraulic engineers. The description of weir formula is used in two ways of

$$Q = qB = (2\sqrt{2g}/3)CBH_o^{3/2}, \tag{23}$$

and

$$Q = qB = (2\sqrt{2g}/3)C_dBh_o^{3/2}, \tag{24}$$

in which B: the weir width, h_o: the upstream depth from the crest, and

C and C_d are discharge coefficients for total head and overflow depth, respectively.

The first treatment is directed to the hydraulic behaviour of C in Eq.(23). C is definitely calculated by Eqs.(21) and (23). In the same manner as Jaeger did, the following dimensionless parameters are introduced, for the convenience of notation,

$$h_c/H_o = K, \quad \text{and} \quad R/H_o = \lambda.$$

Inserting above parameters into Eqs.(21) and (23), the discharge coefficient for total head, C, is expressed in terms of dimensionless parameters of K and λ,

$$C = (3/2)\sqrt{1 - K}(\lambda + K)\log(1 + K/\lambda), \tag{25}$$

and

$$2(1 - K)\{1 + \log(1 + K/\lambda)\} = (\lambda + K)\log(1 + K/\lambda). \tag{26}$$

When the boundary geometry of weir is determined, the discharge coefficient can be calculated by the trial and error method. Fig. 3-1 indicates the theoretical results of C in terms of (R/H_o) calculated by the present approach under the assumption of irrotational curved flow. The curve indicates for small values of (R/H_o) the flow can be more carried over the weir while the opposite behaviour is seen with the decrease of head for a definite geometry of weir. When the curvature of boundary becomes small, (R/H_o) is very large and finally the round crested weir becomes of type of broad crested or long weirs. Consequently, it is readily understood the curve of discharge coefficients tends to 0.577 for very large values of (R/H_o), as derived by the critical depth theory for invicid flows with parallel stream lines.

In the same figure, the theoretical curve of Jaeger's analysis in which the velocity profile was in a form of Eq.(10) is also listed. His calculation based on the Bélanger theorem is derived by

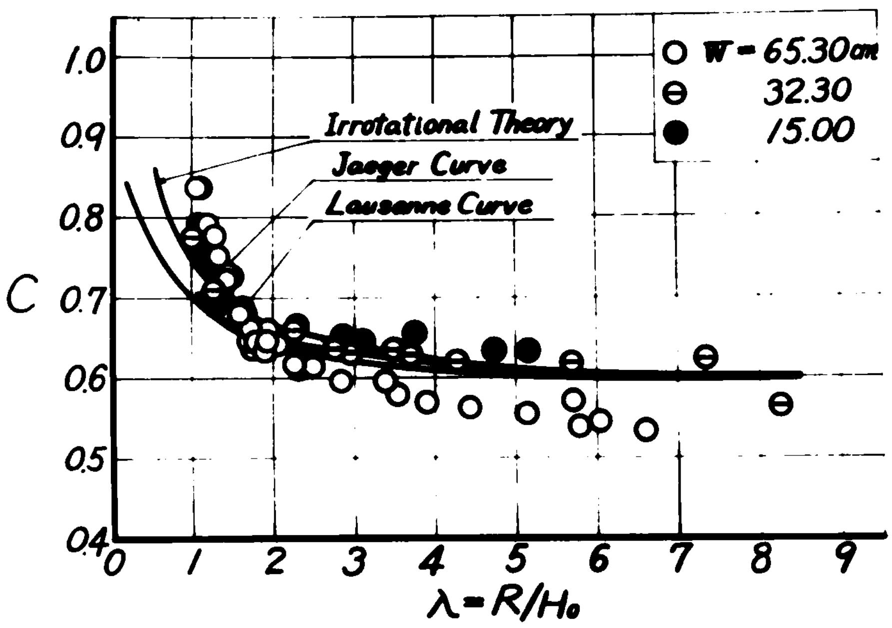

Fig. 3-1 Discharge coefficient of circular weir

$$C = (3/2)\sqrt{1 - K}\{\lambda + 2K - \sqrt{\lambda(\lambda + 2K)}\},$$

and

$$\lambda + 2K = \lambda(2 - \lambda - 4K)^2/(4 - \lambda - 6K)^2.$$

For large values of λ, C also tends to 0.577, which is of equal value derived by the irrotational treatment, whereas for small values of λ, C in the Jaeger curve is less than that derived herein. The Lausanne curve in the figure represents the empirical relationship obtained by the experimental data at the laboratory of University of Lausanne, and it is

$$C = (3/2)(0.385 + 0.085/\lambda - 0.010/\lambda^2). \quad (27)$$

This relationship has a maximum value of C at $\lambda = 4/17$ and thereafter with the increase of head, the discharge coefficient diminishes rapidly.

Experimental data obtained at the Hydraulics Laboratory, Kyoto

University are also plotted in the same figure. The experimental researches have been made with the use of a circular weir of 30 cm in diameter and of 30 cm in width for various discharge from 0 to nearly 30 l/sec. The data are mostly agreed with the theoretical curve of irrotational theory and confirm the validity of present treatment for the flow over a circular weir.

In connection with the location of saddle point predicted uniquely by the theoretical approach, Fig. 3-2 indicates the relationship between discharge and critical depth of the saddle point located at the weir crest for the weir used in the experimental research program. The observed depths are also illustrated in the same figure as a parametric relation of discharge, and it is understood the experimental data confirmed the theoretical prediction of hydraulic behaviours at the saddle point with sufficient accuracies.

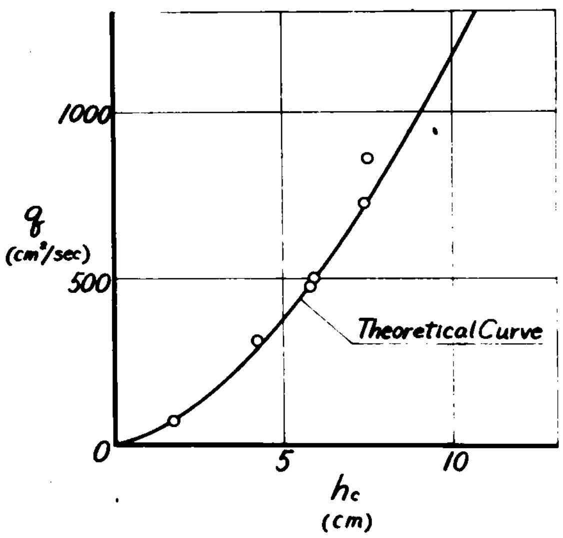

Fig. 3-2 Relationship between critical depth and discharge at weir crest

3 - 2 - 4 Experimental Verification to Theoretical Analysis of Flows over Circular Weir

Theoretical analysis of flows over a round crested weir, as a first approximation by means of the irrotational theory of fluid flows, has been completely treated in the preceding sections and the results of flow characteristics were illustrated in Fig. 3-1 as expressions of the head-discharge relationship. The flow involves a control section at the crest of round crested weir, so that the other hydraulic data necessary to hydraulic design of round crested weirs and other weirs of similar types are also obtained by the theoretical calculation. In this section, for the above design purpose, the experimental verification of flow characteristics will be presented. The whole experimentation was made in a circular weir of 30 cm in diameter at the Hydraulics Laboratory, Kyoto University. Bearing in mind that the singular point is a saddle point for all discharges, the possible surface profiles of flows over a round crested weir as solutions of basic equations of (18) or (19) in 3-2-2 are illustrated in Fig. 3-3. The only transition curve passing through the saddle point, at which the flow changes from subcritical to shooting, is the solution of flows, so that the round crested weir becomes a control structure. Other curves illustrate the water surface profiles produced by other control structures like gate, dam, weir and so on. The transition curve is obtained by numerically integrating Eqs.(18) or (19) in 3-2-2 up- and downstream from the singular point with the definite value of initial surface slope determined by means of the method described in 1-2-4. The bed pressure which is a significant factor to hydraulic design of weirs of this type is also calculated by Eq.(11) in 3-2-2. Figs. 4-4 (1) and (2) are examples of theoretical results for particular discharges. In the same figures, measured surface profiles of water and bed

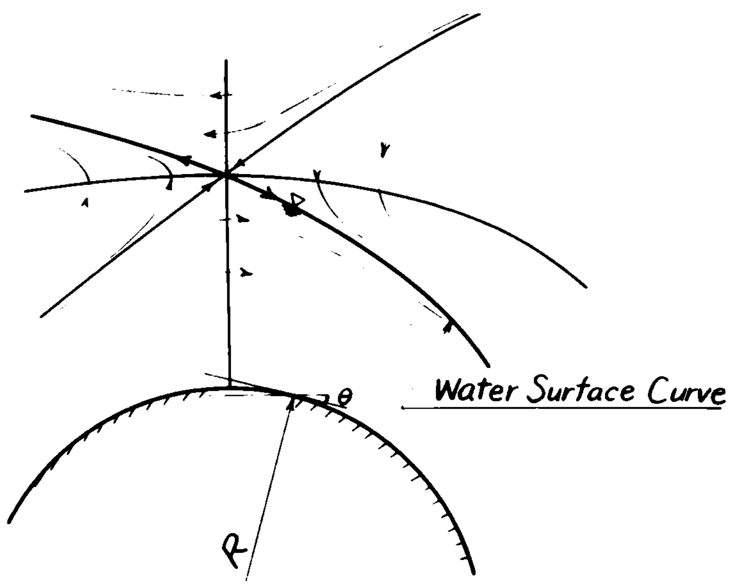

Fig. 3-3 All possible surface profiles of water over round crested weir

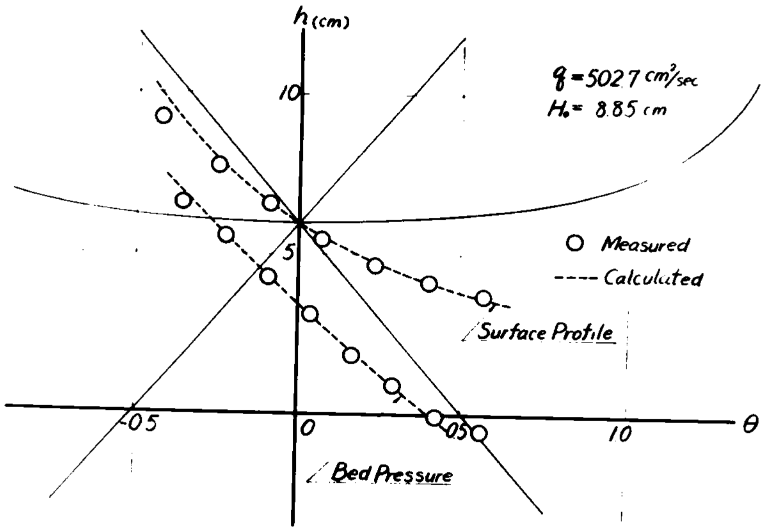

Fig. 3-4 (1) Surface profiles of water and bed pressure along circular weir (q = 502.7 cm^2/sec)

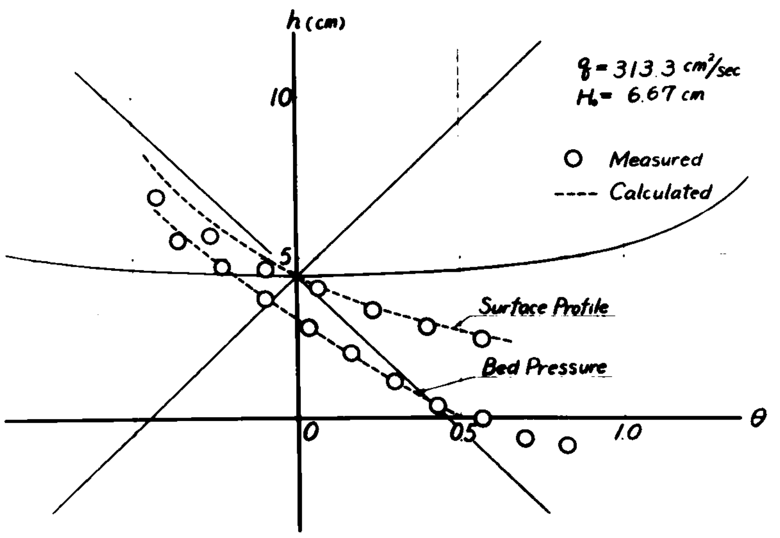

Fig. 3-4 (2) Surface profiles of water and bed pressure along circular weir (q = 313.3 cm^2/sec)

pressure along the curved boundary are also plotted for same discharges. It is evidently understood that the experimental data prove the theoretical method of analysis described herein valid, and rather surprising is that a close agreement is obtained between experimental and theoretical approaches.

For purposes of hydraulic design of round crested weir as well as similar shapes of other hydraulic structures like control structures of overflow spillways, the most important factors are the improvement of discharge capacity from the crest and the prevention of formation in unfavourable pressure decrease, though other factors of vibrations and so on must also be considered. When a round crested weir is planned, the necessary length of solid boundary to regulate the flow direction must be determined by the condition of zero pressure and a longer weir in length than the above described limit is conversely of no need from the pressure condition. When a control structure of overflow spillway is planned in a form of constant curvature, the transition curve must be connected with the crest curve in a part of discharge carrier until the bed pressure becomes zero. The limiting condition of zero pressure over a circular weir is theoretically evaluated by the following method.

At the point where the bed pressure becomes zero, Eq.(11) in 3-2-3 is, in terms of discharge, radius of curvature and depth,

$$2g\cos R^2(R + h)^2\{\log(1 + h/R)\}^2 = (2R + h)q^2. \qquad (28)$$

At the same point, the equation of motion is known as the Bernoulli equation, which is

$$H_o = (u_s{}^2/2g) + h\cos\theta - z. \qquad (29)$$

Introducing the dimensionless parameters of $(R/H_o) = \lambda$, $(h_c/H_o) = K$, $(h/H_o) = K'$ and the relationship of $(z/H_o) = \lambda(1 - \cos\theta)$, eliminating cos from Eqs.(28) and (29), and with the use of

$$(q^2/gH_o{}^3) = (\lambda + K)^3\{\log(1 + K/\lambda)\}^3/\{1 + \log(1 + K/\lambda)\},$$

the final result is

$$\begin{aligned}&(3\lambda^2 + 3\lambda K' + K'^2)/(\lambda + K')^2\{\log(1 + K'/\lambda)\}^2\\ &= 2\lambda^2(1 + \lambda)\{1 + \log(1 + K/\lambda)\}/(\lambda + K)^3\{\log(1 + K/\lambda)\}^3\\ &= 9\lambda^2(1 + \lambda)/4C^2. \end{aligned} \qquad (30)$$

In the above equation, the right term is apparently known by the given channel geometry and the head over a weir crest, so that the critical value of water depth is uniquely calculated by $\lambda = (R/H_o)$. Fig. 3-5 indicates the theoretical relationship, for zero pressure at bed, between the inclination angle of curved boundary and the ratio of R to H_o, and in the same figure, experimental data are also

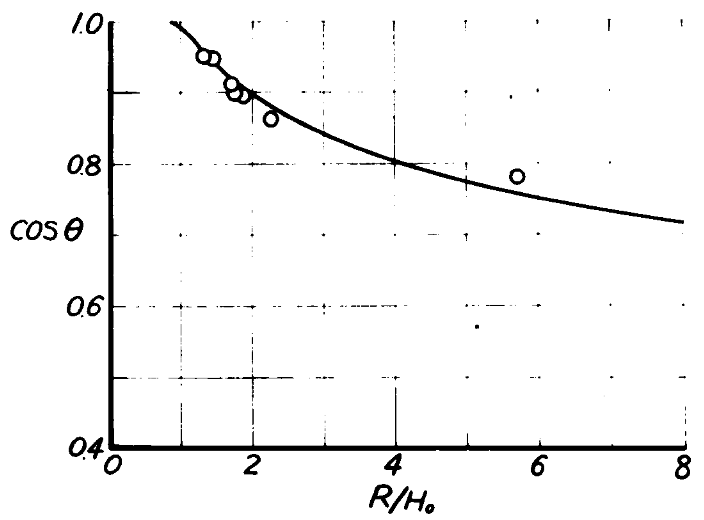

Fig. 3-5 Relationship between $\cos\theta$ and (R/H_0)

plotted. It is understood that a fair agreement between experimental and theoretical data is observed, and consequently, Fig. 3-5 will give available informations of necessary length of curved boundary when the hydraulic design is planned.

1) Jaeger, C., Remarques sur quelques écoulements le long de lits à pents variants graduellement, Schweiz. Bauztg., Vol. 114, No. 20, 1939.

2) Mallet, C., and Gaussens, J., Le siphon d'essais du barrage de l'Oued Fergoug, Terres et Eaux, Vol. 8, 1949.

3) Goldstein, S., Modern Developments in Fluid Mechanics, Vol. 1, Oxford, 1950.

4) Jaeger, C., Engineering Fluid Mechanics, Blackie, London, 1956.

3. Hydraulic Performances of Overflow Spillways as Control Structures

3 - 3 - 1 General Remarks

One of the most popular types of control structures in hydraulic work is known as the overflow spillway, which provides controlled release of surplus water in excess of the reservoir capacity and conveys it to the downstream channel or other watercourses in some other drainage basins. Hydraulic functions of spillways are so wide owing to their diversity in design purposes of flood control, irrigation and power developments. The importance of a safe and adequate design of spillway, however, can not be overemphasized. Many failures of dams have resulted from improperly designed spillways or spillways of inadequate capacity.

Although the general scheme of spillway design is to make successively varying combination of spillway elements until the most economical combination of elements is found, all hydraulic functions of a spillway must be carefully determined. For the determination of spillway capacity, the complete studies of hydraulic behaviours to ensure a safe release of water of design reservoir flood discharge as well as the hydrological studies of basic data are needed, and the details of hydraulic knowledge on pressure distribution and water surface elevation obtained by various methods of theoretical and experimental are also required to make the spillway structure sure.

Spillways are usually composed of five distinctive elements[1]: An entrance structure, a control structure, a discharge carrier, an energy dissipator and an outlet channel. With respect to the present purpose on the hydraulic functions of control structures, the hydraulic performance of control structures of spillway will be of most significance, whereas each of elements is also important so that the spillway itself as a composite structure will serve satisfactorily to its purpose. The control structure followed to the entrance channel can regulate the discharge from the reservoir by means of various types in hydraulic behaviours. Structurally, it may be a controlled or uncontrolled crests , a tube or a pipe. The form of crest shape is usually of a weir type. Normal control structures of overflow spillway for which the water approaches the structure normal to the crests and pass through the structure with no general change in direction are most common.

At uncontrolled spillways, crest shapes are designed to fit the lower nappe of overflow streams from a sharp crested weir during full capacity operation, as one of the basic requirements for spillway design is that the subatmospheric pressure which induces the unfavourable influence to the structure will be excluded. This shape has been studied extensively for various heads under a variety of conditions by many hydraulic engineers, among whom W.P. Creager and J.D. Justin[2], Scinemi[3], and Kindsvater and R.W. Carter[4] are very famous for the study of nappe form and the laboratory of USBR[5] also reported the complete study of coordinates of the lower nappe. For controlled crests, these coordinates produce a surface that will not support a jet of water issuing under gate with partical opening, and to maintain the condition of prevention of subatmospheric pressure, the crest shape is usually flattened to support the jet downstream from the gate. Although a large number of scientists and engineers have sought a comprehensive equation that would describe the form of nappe investigated experimentally, it is still remained unsolved because of being not subjected to the complete mathematical form. Coordinates of crest shapes are usually determined by appropriate mathematical equations like parabola or similar equation to the nappe of sharp crested weir flow. The shape may be designed by a combined form of several mathematical expressions so that the final form becomes similar to the lower nappe. The designed crest shape being approximate to the lower nappe is designated as a standard crest.

The foregoing design procedure is largely based on the pressure requirements for control structure to prevent the structure from instability condition due to locally reduced pressure and possible cavitation, though the hydraulically best spillway design is to select the shape for the highest efficiency of discharge characteristics compatible with the satisfactory pressure condition. As the normal control structure of a spillway is essentially a weir, the discharge characteristics are commonly established by the so-called weir formula as

$$C = q/\{(2\sqrt{2g}/3)H_0^{3/2}\}. \qquad (1)$$

The discharge coefficient C is evidently dependent on approach conditions, shape of crest, downstream submergence, and in general, on any condition that interferes with the free flow of the water. Limiting the problem to the discharge characteristics, the general method of analytical treatment is not existed because the nappe shape can not be expressed in terms of the mathematical form as frequently described. Consequently, the estimation of C for various heads is usually determined by the model tests and with the use of calibrated values of C the discharge over a prototype overflow spillway is predicted. The discharge coefficient usually varies between 0.655 and 0.730 at maximum discharge and 0.505 at minimum discharge with poor entrance, and many empirical relationship for C have been proposed to fit particular shapes of overflow spillways. As examples of discharge characteristics, Fig. 3-6 indicates the relationship between discharge characteristics and head for models of various overflow spillways at TVA projects under the free flow condition presented by K.W. Kirkpatrick[6]. In the figure, C is expressed by

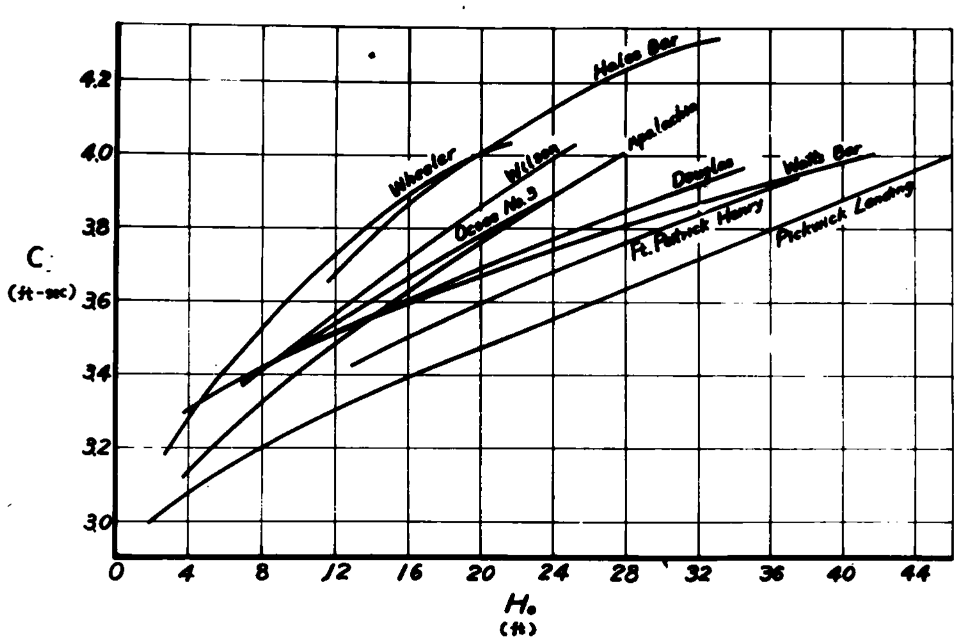

Fig. 3-6 Discharge coefficients of free flow over spillway crests at TVA dams

$q/H_o^{3/2}$ in ft-sec unit, so that the discharge coefficient C in m-sec unit defined by Eq.(1) is obtained by multiplying C in Fig. 3-6 by (1/5.346). The TVA crests all fairly closely approximate the standard curve from the upstream spillway face to a point somewhere downstream from the crest which was determined by the position of the gate seal. Below this latter point, the crest shape was modified to fit the trajectory of jet issuing from the gate when set at a small opening. Among dams listed in the figure, two pairs of these, the Ocoee No. 3 - Apalachia set and the Douglas - Watts Bar set have crest shapes that are identical within the pair. Evidently seen in the figure, the head-discharge relationship for various shapes of overflow crests is dependent on their geometrical shape and no unified relationship of discharge and head for all overflow spillways can be found. In this chapter, the research purpose is to search the unified treatment of head-discharge relationship by the analytical method by means of the transitional characteristics of open channel flows in channel transitions and controls.

3 - 3 - 2 Head-Discharge Relationship of Overflow Spillways

As has been discussed in the preceding section, the discharge characteristics of released water from the reservoir can not be generally unified as a function of upstream total heads for all shapes of overflow spillways, if Eq.(1) is used as the relationship between discharge and head. Many hydraulic literatures have also described the discharge coefficient C would be determined by the scale model tests owing to difficulty of unified description in a mathematical form. Fig. 3-6 are examples of such behaviours.

The flow over a spillway is evidently expressed by

$$(1/q)(d/dx)\int(u^2/2g + p/\rho g + y\cos\theta)u\,dy = \sin\theta - (\tau/\rho gh)(u_b/u_m). \qquad (2)$$

In the foregoing chapter of hydraulics of circular weir, the assumption of constancy in head and $u(R + y)$ was used for the engineering approximation. The analysis of flow behaviours based on the above equation derived by the one dimensional procedure also requires the knowledge of distribution of velocity and pressure in the flow. If the same assumption concerning distribution of velocity and pressure as in the foregoing chapter will be applied, the resulting surface profiles near the spillway crest are expressed by

$$(dh/dx) = f_1(h, x)/f_2(h, x), \qquad (3)$$

in which

$$f_1(h, x) = \sin\theta(1 + h/R) + (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}(dR/dx)\{\log(1 + h/R) - h/R\} - (q^2/C^2h^2R)\{\log(1 + h/R)\}^{-1},$$

and

$$f_2(h, x) = \cos\theta - (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}\{1 + \log(1 + h/R)\},$$

and C is the Chézy roughness. As the designed control structure of spillway must involve the control section for all discharges of released water, both of numerator and denominator are simultaneously zero and the singular point is a saddle point through which the flow changes from tranquil to shooting. In the case of circular weir, the curvature and the head were assumed constant, and consequently the singular point was located at the weir crest, whereas in this case, which concerns with the flow characteristics in more adequate forms, it is observed that the location of saddle point will be transferred to the downstream face from the crest, as the first and second terms in the nominator are usually positive but small in the vicinity of crest, and on the contrary, the last term of frictional resistance negative. When the crest of overflow spillway is of type of constant curvature, it is more definitely understood. After establishing the locations of singular points in the flow, the flow characteristics of control structures of overflow spillway in the immediate vicinity of crest are completely analyzed with the use of Eq.(3). In nearly all overflow spillways, the crest shape is designed to fit the lower nappe of design discharge of free flow from a sharp crested weir, coordinates of shape are different depending on the design discharge, so that the local curvature of crest is not constant. The analysis must be, consequently, solved by the numerical analysis with a tremendous amount of labours.

For the sake of simplicity, the first approximate behaviours of discharge characteristics of control structures of overflow spillway with constant curvature will be treated. Under the assumption of negligible frictional resistance of solid boundary, the overflow crest is evidently the point of control section and the flow characteristics are described in the same expression as shown in the foregoing chapter. The head-discharge relationship of released discharge over a crest is uniquely represented by a single curve as a function of (R/H_o). To verify the above assumption, the discharge relationships at TVA dams of Wilson, Wheeler and Pickwick Landing, which are typical examples of constant curvature[6], will be concerned. The following table indicates the results of analysis based on the foregoing treatment. Column (2) represents values of R at the down-

Relationship between Actual Value of R and Estimated Value by Means of Irrotational Curved Flow Theory at TVA Dams

(1) Dam	(2) R_{ac} (ft)	(3) H_o (ft)	(4) C ft-sec	(5) C m-sec	(6) R/H_o	(7) R_{est}
Pickwick Landing	45.50	12	3.31	0.618	4.25	51.0
		16	3.39	0.634	3.20	51.2
		20	3.47	0.650	2.46	49.2
		24	3.55	0.664	2.00	48.0
		28	3.64	0.680	1.65	46.2
		32	3.72	0.695	1.45	46.4
		36	3.80	0.711	1.26	45.4
		40	3.89	0.727	1.13	45.2
		44	3.97	0.742	1.03	45.3
Wilson	21.30	8	3.41	0.638	3.02	24.1
		12	3.57	0.668	1.88	22.6
		16	3.72	0.696	1.44	23.1
		20	3.86	0.722	1.17	23.4
		24	3.99	0.747	0.98	23.5
Wheeler	16.00	4	3.28	0.612	4.85	19.4
		8	3.53	0.660	2.11	17.9
		12	3.73	0.697	1.43	17.2

		16	3.89	0.727	1.12	17.9
		20	4.01	0.749	0.96	19.2

stream face. If the present approximation will be satisfactory as the engineering purpose, R must be the local value at the spillway crest. However, in usual design procedures, the spillway shape is expressed by combined curves connected at the crest to fit only the pressure requirement, and consequently the local curvature is commonly discontinuous at the crest and other point where curves are connected together. Bearing in mind that the control section actually will be observed at the downstream face near the crest, as seen in Fig.(3), R is represented by the value at the downstream face. Columns (3) and (4) are calculated from data indicated in Fig. 3-6 in ft-sec unit. Discharge coefficients C in column (5) which is defined by Eq.(1) in the preceding section are transformed to values in m-sec unit, so that with the use of the relationship of Eqs.(25) and (26), R/H_0 is also calculated, and finally the value of curvature of $(1/R)$ is estimated for various upstream heads. With sufficient accuracies, the estimated values of R are agreed with the actual values of R at these dams. It , therefore, indicates that the flow near the crest of spillway is approximated by a curvilinear motion. The discharge characteristics of flows at the overflow spillway with constant curvature, therefore, can be predicted in terms of (R/H_0) without conducting model experiments, and it is rather surprising that all curves of discharge relationships to head at Wilson, Wheeler and Pickwick Landing Dams become a single parameter curve, whereas in the usual representations of discharge coefficients, these curves can never be expressed by a curve. Fig. 3-7 also is a graphical representation of the head-discharge relationship. In the figure, the data of flow characteristics at the Suiho Dam in

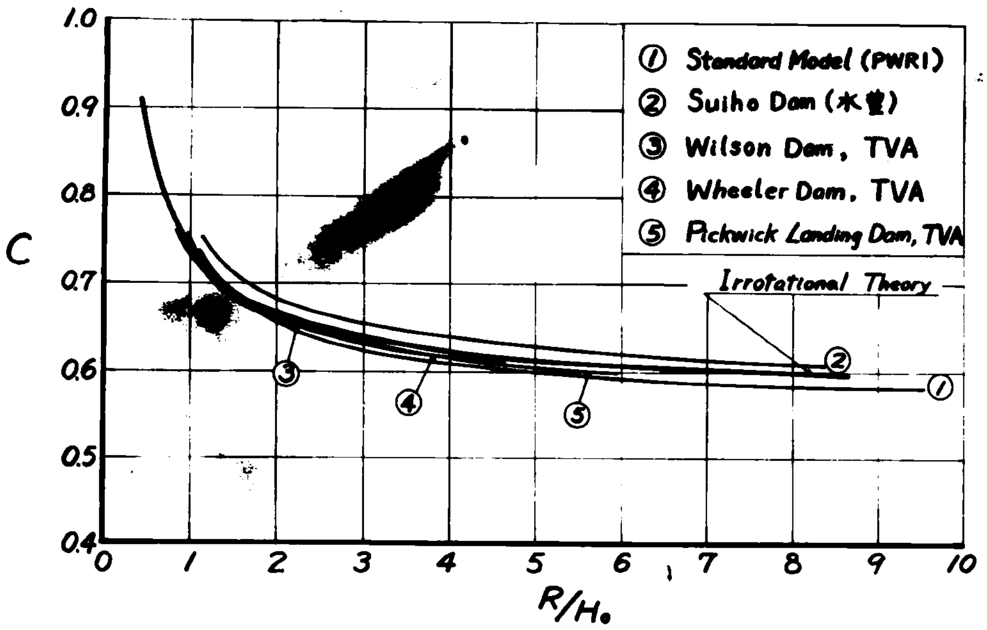

Fig. 3-7 Discharge coefficients of free flow at TVA dams and Suiho dam by means of irrotational curved flow theory

North Korea obtained by the model study at the Public Works Research Institute, Ministry of Construction (formerly Ministry of the Interior)[7] are plotted. The crest shape is similar to the standard crest and somewhat modified. The result also indicates a good agreement.

The same approximate treatment will be applied to other dams of variable curvature at TVA projects, Fort Patrick Henry, Hales, Apalachia and Ocoee No. 3 Dams. Among these dams, the dimensions are identical at the Apalachia and Ocoee No. 3 Dams. The local curvature is variable, in this case, so that the tremendous numerical calculations are needed to find the complete behaviours of discharge characteristics. The comparison of actual value of R at these dams to the estimated value of R by means of the foregoing approximate analysis will be made, and the resulting table is in the following.

Relationship between Actual R and Estimated Value of R at TVA Dams of Variable Curvatures

(1) Dam	(2) Distance (ft)		(3) R_{ac} (ft)	(4) H_o (ft)	(5) C ft-sec	(6) R_{est}
Fort Patrick Henry	Up-stream	-2	22.0	16	3.51	34.3
		-1	43.9	20	3.60	34.5
	Down-stream	0	0	24	3.68	34.6
		1	24.0	28	3.76	34.6
		2	27.7	32	3.85	34.8
		3	30.5	36	3.91	34.8
		4	32.8			
Hales Bar	Up-stream	-2	6.1	10	3.59	18.7
		-1	11.6	12	3.68	18.6
	Down-stream	0	0	16	3.86	18.4
		1	14.2	20	4.01	18.6
		2	16.6	24	4.14	19.2
		3	18.7	28	4.25	20.2
		4	20.6	32	4.31	21.6
Apalachia	Up-stream	-2	9.6	8	3.33	31.2
		-1	18.9	12	3.49	29.1
	Down-stream	0	34.3	16	3.63	31.6
		1	34.7	20	3.77	26.4
		2	35.3	24	3.89	26.4
		3	36.0			
Ocoee No. 3	Up-stream	-2	9.6	6	3.35	22.2
		-1	18.9	8	3.41	24.4

	Down-stream	0	0	12	3.55	24.4
				16	3.67	25.1
		1	34.7	20	3.78	25.2
		2	35.3	24	3.88	25.9
		3	36.0			

Evidently seen in the table, values of R estimated by the approximate treatment are nearly constant for a wide range of upstream heads, whereas the local curvature changes from point to point in these dams. Of special evidence is that the estimation indicates the actual value in engineering accuracy and the validity of the present treatment for determination of hydraulic performance of overflow spillway will be expected.

In reality, for hydraulic design of crest shape in overflow spillways, the common procedure is to use the Creager curve for free flow nappe, which can not be explicitely expressed in a mathematical form, so that local curvatures may become discontinuous at the control section as described, and a suitable approximation provides the crest curve to fit the pressure condition. For examples, values of R at dams of Fort Patrick Henry, Hales Bar, Apalachia and Ocoee No. 3 are suddenly changed at the crest as seen in the table, and these design procedures for crest shape are widely observed in almost all projects. If the continuous curve in curvature is used for the design purpose of control structures in overflow spillway, the head-discharge relationship by the spillway will be theoretically established by means of the present approach, and this requirement compatible with the pressure requirement is possibly needed for pertinent hydraulic design of spillway structures.

1) Bureau of Reclamation, Treatise on Dams, Chapter 12, Spillway, Vol. 10, Design and Construction, Design Supplement No. 2 to Part 2 Engineering Design, U.S. Department of the Interior, 1952.

2) Creager, W.P., and Justin, J.D., Hydroelectric Handbook, John Wiley, New York, 2 nd Edi., 1950.

3) Scinemi, E., Sulla relazione che intercede fra gli sccui osservati nelle opere idrauliche originali e nei modelli, Energ. ellettr., 1939.

4) Kindsvater, C.E., and Carter, R.W., Discharge Characteristics of Rectangular Thin-Plate Weirs, Jour. Hydraulics Division, Proc. ASCE, HY 6, Dec. 1957.

5) Studies of Crests for Overfall Dams, Bull. 3, Boulder Canyon Project, Final Reports, Bureau of Reclamation, U.S. Department of the Interior, Denver, Colo., 1948.

6) Kirkpatrick, K.W., Discharge Coefficients for Spillways at TVA Dams, Trans. ASCE, 1957.

7) Takenouchi, T., Hydraulic Medel Tests for Influences of Overflow Spillway Shapes to Discharge Coefficients, Reports of Civil Engineering Experiment Station, Ministry of the Interior, (now Public Works Research Institute, Ministry of Construction), Vol. 73, Nov. 1943 (in Japanese).

4. Sharp Crested Weir and Sill as Control Structures in Hydraulic Works

3 - 4 - 1 Basic Consideration

For many years since the initiation of hydraulic research, the hydraulics of flows over a sharp crested weir has been widely studied, in connection with flow measurements required for various fields of practical engineering, and consequently a large number of scientists and engineers have sought a comprehensive equation that would be expressible for the discharge characteristics of sharp crested weir for a full range of fluid, flow and geometric variables. Nevertheless, the complete mathematical description of flow characteristics has not yet obtained owing to the great complexity in flow behaviours characterized by free, curvilinear flow and other combined influences of several properties, so that there are many empirical formulas of weir characteristics presented by many engineers through their own experimental researches as seen in hydraulic literatures.

The most popular formulas of discharge equation over a sharp crested weir are of type expressed by

$$q = (2\sqrt{2g}/3)C_d h_o^{3/2}, \qquad (1)$$

and

$$q = (2\sqrt{2g}/3)CH_o^{3/2}, \qquad (2)$$

in which h_o: the upstream depth from the weir crest and H_o: the total head from the same level.

The hydraulic treatment to describe the head-discharge relationship by a sharp crested weir is divided into two ways of analytical and dimensional analysis procedures. The first method initiated by G. Poleni[1)] in 1717 is to derive the discharge equation by integration of local velocity over a whole flow area at the point of weir. The basic flow patten, however, is not subjected by a complete mathematical description, and the method may be related to a quasi-rational analysis in which the weir is described as limiting example of the two dimensional orifice. As the flow pattern at the crest section is not distinctively described, so the method is evidently approximate and the discharge coefficients defined by Eqs. (1) and (2) involve all hydraulic factors which can not be theoretically obtained. Among many empirical formulas of head-discharge relationship, the following relationships are still famous, though the Francis formula[2] is frequently used for the rough estimation of discharge owing to its simplicity in mathematical form.

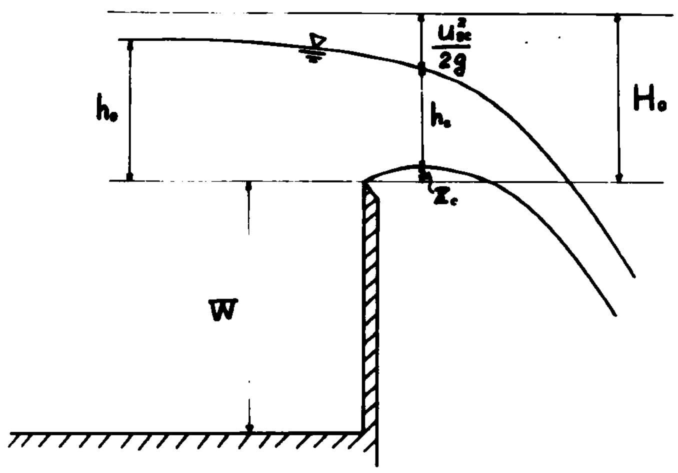

Fig. 3-8 Flow over a Sharp Crested Weir

$$q = (0.405 + 0.003/h_o)\{1 + 0.55h_o^2/(h_o + W)^2\}\sqrt{2g}h_o^{3/2}, \quad (3)$$

in which W is the height of weir. It is famous as the Bazin formula[3] published in 1888, and the dimensional term $0.003/h_o$ is introduced to compensate for low head effects attributed to the surface tension and viscosity.

Another famous equation is the Rehbock formula[4] proposed from 1911 to 1928, and the most pertinent type of equation presented is

$$q = (0.605 + 0.08h_o/W + 1/1000h_o)(2\sqrt{2g}/3)h_o^{3/2}. \quad (4)$$

If the capillary and viscous influences being remarkable at very low heads are ignored in Eqs.(3) and (4), the expression of discharge coefficients in the Bazin and Rehbock formulas become

$$C_d = 0.608 + 0.334h_o^2/(h_o + W)^2, \tag{5}$$

and

$$C_d = 0.605 + 0.08h_o/W, \tag{6}$$

which are plotted in Fig. 3-9. A somewhat revised formula of Rehbock is published by Ippen[5], and it is

$$C_d = 0.611 + 0.08h_o/W. \tag{7}$$

At low heads of flows, two curves indicate nearly the same tendency, whereas at high heads, for which the flow over a weir tends to the free overfall at a control sill in its flow behaviours, they evidently differ. The limiting value of the former is 0.942, while the latter equation yields the value of C_d infinite.

Recently, F. Paderi[6] modified the discharge equation which is of type of Eq.(1) with the use of the Francis formula as the basic discharge equation over a sharp crested weir, to obtain the complete description of head-discharge relationship for a full range of upstream heads, $h_o/(h_o + W) = 0 - 1$. The resulting equation is

$$C_d = (3/2)\{0.707 - 0.302\sqrt{1 - h_o^2/(h_o + W)^2}\}, \tag{8}$$

and also plotted in Fig. 3-9.

The foregoing treatment is derived by the rather empirical method. The flow over a sharp crested weir is evidently curvilinear and free, and the further approximate procedure taking account of such flow behaviours was presented by W, Bleines[7] in 1953. The basic assumption of his analysis is that the velocity distribution of flows at the section of maximum height of lower nappe is described in terms of the quadratic expression of dimensionless depth.

Putting the pressure is zero at upper and lower nappes, and assuming the value of surface slope and the ratio of critical depth to local radius of curvature are constant, the head-discharge relationship of a sharp crested weir is obtained in the following, with the use of the Bélanger theorem for maximum discharge.

$$C_d = 0.644\{1 + 0.350C_d{}^2h_o{}^2/(h_o + W)^2\}^{3/2}. \qquad (9)$$

Another theoretical treatment of discharge characteristics over a sharp crested weir was made by R. von Mises[8] in 1917, and

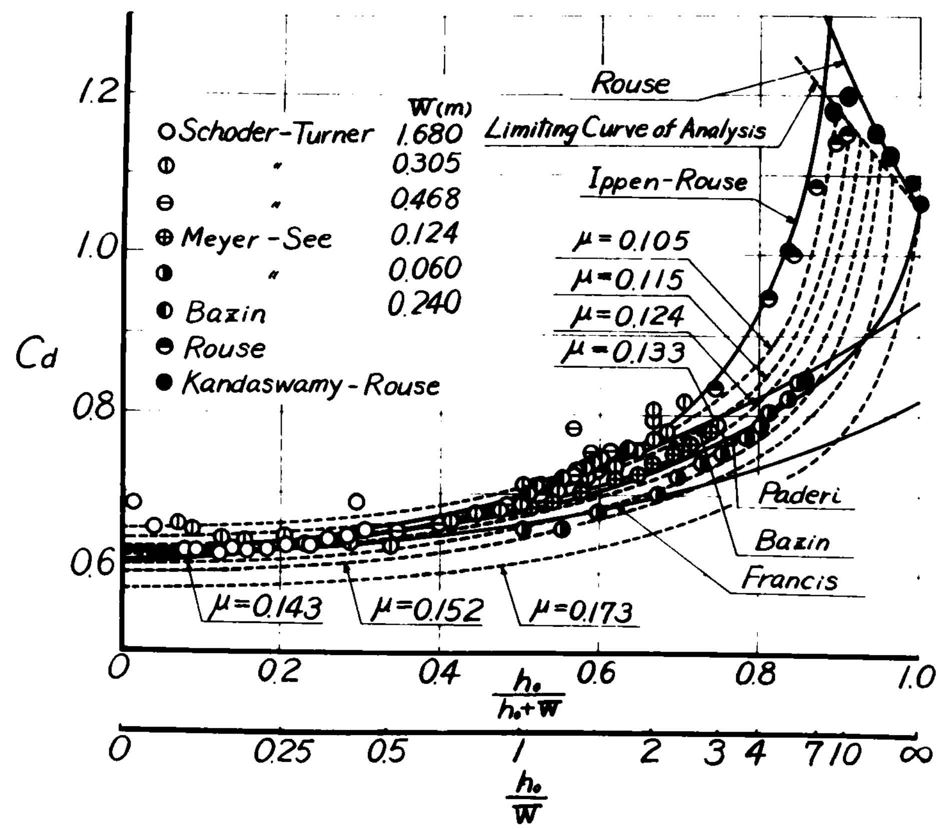

Fig. 3-9 Discharge coefficients of flows over a sharp crested weir in terms of upstream depth

he noted that the values for a symmetrical slot in a normal plate at the end of a conduit would duplicate the trend of Rehbock formula very closely over a considerable range if the ratios of slot dimension to conduit dimension were used to prepare the ratio

$h_o/(h_o + W)$.

As has been described in the foregoing, the physics of curved flow over a weir can not be described in a complete mathematical form by the present hydraulic knowledge, so that the other procedure of analysis as the dimensional analysis will be arised. Although the method will indicate the most direct solution for the basic flow pattern in a form of desirable discharge equation, the dynamics of flow behaviours at the weir evidently can not be obtained. In 1957, C.E. Kindsvater and R.W. Carter[9)] used this procedure with experimental data of many engineers as well as their own. Of most significance in their dimensional analysis is the introduction of effective head and weir width which may represent the combined effects of several phenomena attributed to viscosity and surface tension. As seen in Fig. 3-9, the measured discharge coefficients become larger than curves of empirical relationship of many engineers, owing to the appreciable magnitude in capillary and viscous influences. By means of the results of Kindsvater and Carter, the discharge equation of (1) in terms of effective values of head and weir width has been represented by a single straight line for relatively low heads. The supplementary values should be added to obtain the effective head are changed from 0.09 to 0.36 cm by particular sharp crested weirs used in various experimental studies. The comprehensive relationship of discharge and head as a combined feature of flow behaviours in the light of past experimental data and modern knowledge in theoretical hydraulics, therefore, can not be expected by his procedure, though at very low heads, the discharge equation will be the better expression of head-discharge relationship. The final stage to make the clear formulation of discharge characteristics stands still far away. The measurements of basic hydraulic quantities in velocity, pressure and water stage are urgently needed, with the advance in theoretical hydraulics, to make the successive treatment.

In this chapter, an attempt to derive the theoretical determination of the discharge-head relationship of flows over a sharp crested weir by means of the transitional characteristics of open channel flows will be described, as an example of application of the basic theorem for the simultaneity of maximum discharge and minimum head proved in 1-2-5 to the hydraulic performance of control structures. The knowledges of distribution of velocity and pressure in flow are required, when the usual one dimensional procedure of analysis is used to the flow. The available data for the analysis have not been obtained, despite of long continuation of experimental works on this problem, so that the basic assumption is putted on that the flow is irrotational and the velocity distribution is expressible by Eq.(9) in 3-2-2, as the flow characteristics will be observed rapidly varied within a short distance from the weir crest. The hydraulic treatment thus becomes substantially equal to the flow over a round crested weir, though the condition in pressure distribution in the former case is determined by both upper and lower nappes. The flow direction over a round crested weir is regulated by the solid boundary, while the released water from a sharp crested weir is free and can not be regulated by other geometric variables. It is also another difficulty of theoretical analysis for the discharge behaviours of flows, which will be understood in the later section. Consequently, the present analysis is limited to an attempt to obtain the universal law in discharge equation.

In the above description of hydraulic behaviours of flows over a sharp crested weir, the limiting case in which the ratio of $h_o/(h_o + W) = 1$ or $(h_o/W) = \infty$ is not included. When the Rehbock and Ippen-Rouse formulas are applied to this case, the discharge coefficients become infinitely large, while the actual value of C_d is evidently definite. This problem is known as the free overfall from a sill and the phenomenon is frequently observed at the end terminal of broad crested and long weirs as well as other control structures of similar types.

The hydraulics of free overfall in connection with the determination of the brinking depth as a device for discharge measurement was initiated by P. Böss[10] in 1929. Under the assumption of linear decrease in pressure distribution in the immediate vicinity of terminal, he obtained the discharge relationship over a sill. H. Rouse[11], in 1936, also studied the discharge characteristics of free overfall with the use of discharge equation of Weisbach and obtained the head-discharge relationship for very low weirs and sills in a form of

$$C_d = 1.061(1 + h_o/W)^{3/2}. \qquad (10)$$

For deduction of the above formula, he used the critical regime of flow would prevail upstream from sills of small but finite height as Böss noted. The basic flow characteristics, however, are not concerned by the mathematical expression, with experimental verification, so that the hydraulic significance in the analysis will not be involved. With respect to the ratio of brinking depth h_b to critical depth of parallel flow for the same rate of discharge, Rouse obtained the value of ratio was 0.715 experimentally. A. Craya[12], in 1948, calculated approximately the same ratio in the case of parallel flows even at a terminal sill by integrating the velocity profile derived by the Bernoulli theorem throughout the whole flow area at an end section and obtained the value of 0.650. In the same year, C. Jaeger[13] also calculated this ratio by means of the approximate theory of curved flows with supplementary experimental data of Fawer and indicated the ratio was 0.720. The most significant feature of flows over a sill and a broad crested weir in the immediate neighbourhood of a free overfall was first described by A. Fathy and M.S. Amin[14]. When the flow over a broad crested weir is considered, the hydrostatic pressure prevails in the almost all regions of the crest as seen in Fig. 1-6, whereas the non-hydrostatic pressure owing to the curvilinear motion in fluid flows becomes extreme at the free overfall. The classical treatment of gradually varied flows in open channels like the Bresse equation, therefore, is invalid near the end portion. If the successful treatment will be anticipated, the introduction of pressure coefficient of Jaeger as described in 1-1-3 must be involved. Furthermore, the x-wise distribution of α and λ in the energy equation of gradually varied flows also is required so that the complete behaviours of flows are evaluated. The analysis of Fathy and Amin is treated with the possibility to reveal the flow behaviours of free overfall. Fig. 3-10 indicates some examples of the x-wise distribution of α and λ numerically calculated by experimental data obtained in the long weir of 4.0 m in length and 0.4 m in width, in which the weir sill is elevated by 0.105 m from the original bed, at the Hydraulics Laboratory, Kyoto University. Evidently seen in the figure, the non-hydrostatic pressure prevails at the free overfall, but the definite relationship can not be analytically obtained. Fig. 3-11 indicates the ratio of local radius of curvatue which is calculated by the use of the relationship of

$$-(1/\rho)(\partial p/\partial y) = g\cos\theta - u^2/(R + y),$$

to that of R_{cal} estimated by the irrotational theory described in the foregoing chapter at the free overfall. The value of R is not constant but large in the middle portion of flow and small in the upper and lower portions. However, it will be seen the average value

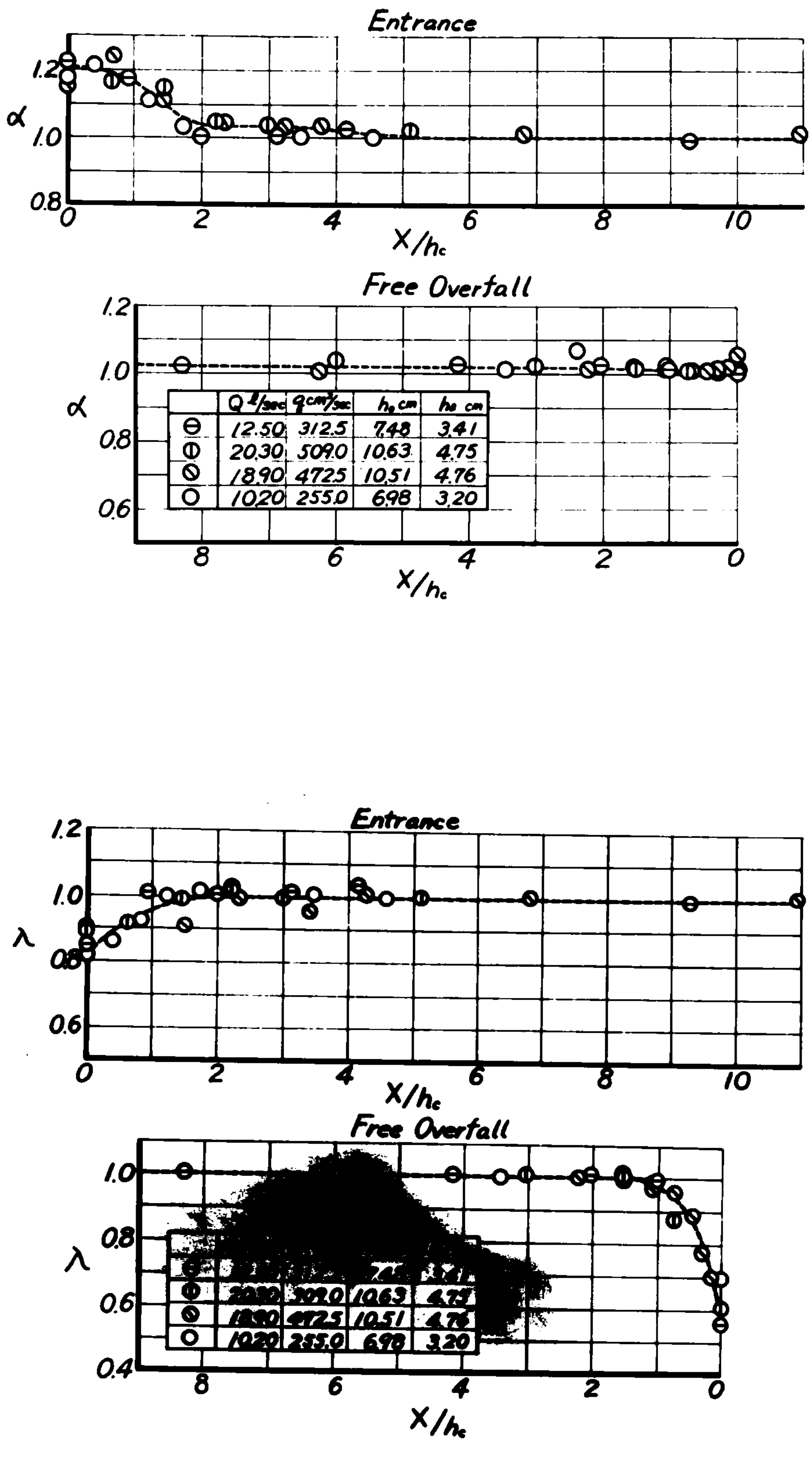

Fig. 3-10 X-wise distribution of α and λ of flows over long weir

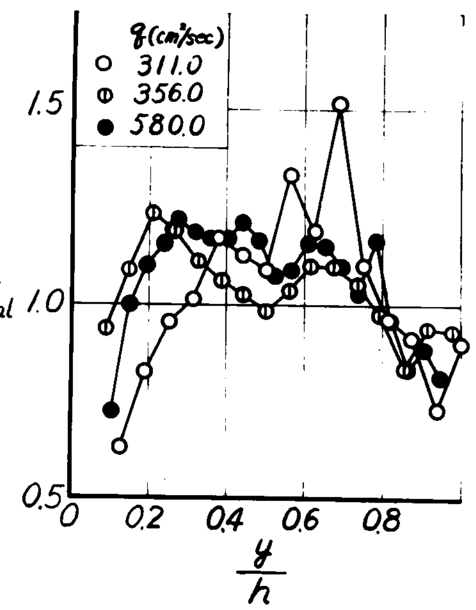

Fig. 3-11 Relationship between R and R_{cal} at free overfall

is nearly equivalent to R_{cal}, so that the irrotational theory will be available for the approximate analysis of the free overfall if the one dimensional procedure is used.

The basic flow characteristics at a sharp crested weir and a sill by means of the irrotational theory of curved flows and the head-discharge relationship obtained by the present analysis will be first treated. The next section concerns with the comparison of various empirical formulas of discharge equation and is followed by some supplementary characteristics of free overfall in the last section. The problems described herein are those of the oldest hydraulic research projects since the birth of classical hydraulics. Nevertheless, owing to insufficiency of basic mathematical knowledge to describe the flow pattern in a complete form of mathematical expression and of experimental measurements which are satisfactory to verify the basic principles of flow, the final formulation of hydraylic characteristics of flows over a sharp crested weir and a sill as control and transition structures will not be established.

3 - 4 - 2 Basic Flow Characteristics of Free Flows over Sharp Crested Weir and Sill

(a) Basic Characteristics of Free Flows over Sharp Crested Weir

Under the assumption of irrotational motion of water flow, the velocity and pressure distributions are expressible by Eqs.(9) and (11) in 3-2-2. Being different from the flow over a round crested weir, the flow direction from a sharp crested weir is free depending on the flow characteristics, so that the condition that the pressure is zero along the lower nappe makes a virtual boundary and it is, from Eq.(11) in 3-2-2,

$$\cos\theta = (q^2/2g)(2R + h)(R + h)^{-2}R^{-2}\{\log(1 + h/R)\}^{-2}. \qquad (11)$$

Consequently, the pressure distribution in the free flow will be approximately expressed by

$$p/\rho g = (q^2/2gR^2)(R + h)^{-2}\{\log(1 + h/R)\}^{-2}\cdot y(h - y)\{3R^2 + 2Rh + (2R + h)y\}/(R + y)^2 . \qquad (12)$$

With the use of the relationships for velocity and pressure distributions described, the one dimensional equation of energy approach along the virtual boundary is derived as follows by means of the equation described in 1-1-4.

$$H_o = (u_s{}^2/2g) + h\cos\theta + z = (u_b{}^2/2g) + z, \qquad (13)$$

in which the origin of coordinate system is selected at the weir crest and z is measured from the weir crest. Eq.(13) is apparently the Bernoulli equation and indicates the lower velocity is larger in its magnitude than that of the upper nappe. The surface profile equation is then derived by differentiating Eq.(13) with respect to x, and the resulting equation is

$$(dh/dx) = f_1(h, x)/f_2(h, x) \qquad (14)$$

where

$$f_1(h, x) = -(dz/dx) + h\sin\theta(d\theta/dx) + (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}\{\log(1 + h/R) - h/R\}(dR/dx),$$

and

$$f_2(h, x) = \cos\theta - (q^2/g)(R + h)^{-3}\{\log(1 + h/R)\}^{-3}\{1 + \log(1 + h/R)\}.$$

As the approximation, $(dz/dx) = -\sin\theta$, $(d\theta/dx) = (1/R)$, and $(dR/dx) \doteq -\tan\theta$, so that Eq.(14) is transformed into

$$(dh/dx) = f_1(h, x)/f_2(h, x), \tag{15}$$

where

$$f_1(h, x) = \sin\theta h\{(5R^2 + 4Rh + h^2)\log(1 + h/R) + 2R^2\},$$

and

$$f_2(h, x) = R(2R + h)(R + h)\log(1 + h/R)\Big[\cos\theta - (q^2/g)(R + h)^{-3} \{\log(1 + h/R)\}^{-3}\{1 + \log(1 + h/R)\}\Big].$$

It is consequently understood from Eq.(15) that the singular point is located at the maximum height of lower nappe. The sharp crested weir serves as a control structure which can uniquely determine the head-discharge characteristics for particular rate of discharge, and for this aim, the singular point must be classified as a saddle point. In fact, the characteristic equation of the linearized version of Eq.(15) represents the singular point as a saddle point. Therefore, the relationship between discharge and head of flows over a weir can be calculated by the simultaneity theorem of maximum discharge and minimum energy, which has been proved in the foregoing part.

The discharge equation is then expressed in an equivalent form of that of flows over a round crested weir, if the equation is represented in terms of R, h, and θ . However, in this case, these values except h are not determined by the direct measurements, so that the total head above the weir crest line will be used. Introducing the following dimensionless parameters of local radius of curvature, water depth and lift of lower nappe from the crest,

$$R/H_o = \lambda, \quad h/H_o = K, \text{ and } \quad z/H_o = \mu ,$$

the pressure condition of Eq.(11) becomes

$$(2\lambda + K)\mu = (1 - K)(2\lambda + K) - \lambda^2, \tag{16}$$

the discharge relationship derived by the Bélanger-Böss theorems, which indicate the denominator term of Eq.(15), is

$$2\{1 + \log(1 + K/\lambda)\}\mu = 2(1 - K) - (\lambda + 3K - 2)\log(1 + K/\lambda), \tag{17}$$

and the discharge coefficient C in terms of the total head is

$$C = (3/2)\sqrt{1-\mu} - K(\lambda + K)\log(1 + K/\lambda)$$
$$= (3/2)\sqrt{1-\mu}\,\lambda\log(1 + K/\lambda). \tag{18}$$

Unknown values in Eqs.(16) - (18) are K, λ, C and μ, whereas numbers of given equations to describe the flow characteristics are three, and therefore the solution will be obtained by a parametric expression of one independent variable. Putting this parameter by μ, the resulting behaviours of discharge characteristics over a sharp crested weir are solved by the trial and error method and illustrated in Fig. 3-12. Apparently seen in the figure, the decrease of discharge coefficient is resulted from the strong contraction of nappe. If the value of μ is a constant as Rehbock noted through his experimental study, the discharge coefficient is uniquely determined. The physical relationships to describe the flow pattern over a sharp crested weir are insufficient in numbers to determine the behaviours of dependent

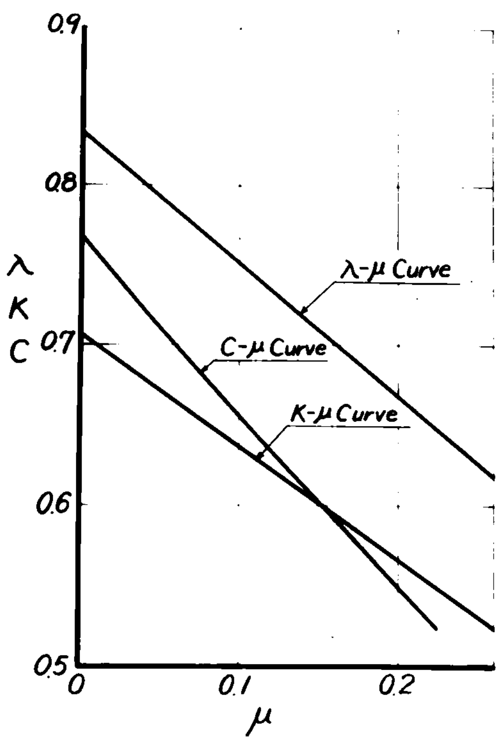

Fig. 3-12 Discharge coefficient C of flows over a sharp crested weir

variables. Although it is expected another principle will be existed, the present knowledge can not establish the relationship with satisfactory degree in the engineering purpose.

(b) Relationship between Head and Discharge in Flows over Sharp Crested Weir and Sill

The most popular form of discharge equation of flows over a sharp crested weir and a sill is expressed by Eq.(1) of 3-4-1. The transformation of discharge coefficient from C to C_d is made by the following procedure. In the upstream approach channel in which the flow is tranquil and assumed uniform in velocity as the first approximation. The total head from the level of crest line is then, assuming $\alpha = 1$,

$$H_o = h_o + (q^2/2g)(h_o + W)^{-2}. \tag{19}$$

Introducing the dimensionless overflow depth $\pi = (h_o/H_o)$, Eq.(19) becomes

$$C_d = (3/2)(1 - \pi)^{1/2}\pi^{-1/2}(1 + W/h_o), \tag{20}$$

and it represents that the discharge coefficient in terms of the overflow depth C_d is estimated by the ratio of overflow depth and weir height as seen in many empirical formulas through the parametric expression of (h_o/H_o). Another relationship between C and C_d is obtained by combining two equations of (1) and (2) in 3-4-1, and the resulting expression is

$$C_d = C\,\pi^{-3/2}. \tag{21}$$

With the use of Eqs.(20) and (21), the usual description for discharge coefficient C_d is calculated as a function of $h_o/(h_o + W)$ in terms of a parametric solution of (z_c/H_o). Fig. 3-9 indicates these solutions for C_d. Parametric values of μ = 0.173, 0.152, 0.143, 0.133, 0.124, 0.115 and 0.105 represent that the discharge coefficient C_d, when the approaching velocity in the upstream reservoir is zero, are 0.578, 0.600, 0.610, 0.620, 0.630, 0.640 and 0.650 respectively. Empirical curves proposed by Francis, Bazin, Ippen-Rouse, and Paderi are also plotted in the same figure. Rehbock noted through his experimental study that the dimensionless critical depth (h_c/H_o) was nearly 0.66 and (z_c/H_o) 0.11 without the change of upstream head. If the ratio of z_c to H_o is 0.11, the value of critical depth becomes 0.63 by the present analysis, and consequently the head-discharge relationship is theoretically calculated. The curve for $\mu = 0.11$ is similar to the formulas of Bazin, Ippen-Rouse, and Rehbock in a range of $h_o/(h_o + W) = 0.3 - 0.7$ and for small values it becomes larger than other empirical curves, while it is smaller than curves of Ippen-Rouse and Rehbock for larger values. In fact, a large number of experimental data presented by Schoder-Turner[15], Meyer-See[15], Bazin[3] and Rouse[5] for almost all values of $h_o/(h_o + W)$ indicate z_c is not constant in a full range of depth but a variable depending on the flow and geometric characteristics, and empirical formulas also support the above description. Of quite interesting indication in experimental data is that the behaviour of C_d for a particular experimentation is closely similar to the theoretical curve of irrotational theory. A functional relationship between (z_c/H_o) and $h_o/(h_o + W)$ will be therefore expected to exist, but the present knowledge of hydraulics can not make the resulting relationship clear. After the success of precise measurements of velocity and pressure distributions will be finished, the present treatment of analysis will be surely advanced.

For large values of $h_o/(h_o + W)$, which indicate the increase of overflow depth for a definite weir, the hydraulic behaviours of flow over a sharp crested weir will be classified as those of flow over a sill or a broad crested weir. Concerning the flow characteristics of free overfall at the sill, Rouse[5] obtained the discharge equation expressed by Eq.(10) in 3-4-1, with the use of suggestion of Böss that the flow over a weir of small but finite height becomes critical upstream from the weir where the hydrostatic pressure prevails in the flow. The curve of Rouse for the discharge characteristics of sills with experimental data is also plotted in the same figure. These experimental data of Rouse and Kandaswamy-Rouse[16] are very closely agreed with empirical curves of Rouse and Ippen-Rouse. Aside from the practical purpose, the suggestion of Böss can not be verified by the theoretical consideration in hydraulics.

In the foregoing discussion of flow behaviours, the limiting condition of the present approach has not been referred. The dimensionless overflow depth varies from 1 to 0.667, as the value of unity indicates the condition of no approaching velocity and 0.667 the critical condition, over which the upstream flow is shooting and the weir is then not a control structure. If the dimensionless height of lower nappe at the control section is assumed, the relationship between $h_o/(h_o + W)$ and C_d can be established by Eqs.(10) and (11), when the upstream flow is critical. The curve of C_d and $h_o/(h_o + W)$ for $\pi = 0.667$ is a limiting condition of the present analysis, and plotted in Fig. 3-9. Evidently, for a free overfall, $C_d = 1.061$, and with the increase of weir height C_d gradually increases. It is also understood from the figure that the value of μ is never constant, as the flow characteristics change arbitrarily. It is rather surprising that experimental data of Rouse and Kandaswamy-Rouse agree closely with the limiting curve of the irrotational theory.

The present analysis describes that the discharge characteristics of flows over a sharp crested weir and a sill will be possibly made clear by the theoretical procedure of hydraulic treatment by means of the transitional characteristics at the singular point if μ is determined as a variable of flow properties. It has already indicated that the empirical conclusion of Rehbock on constancy of μ was invalid, and with respect to the free overfall, Rouse obtained the form of lower nappe for various flow conditions. Nevertheless, the experimental results can not provide the available informations to the theoretical procedure for establishment of discharge characteristics.

3 - 4 - 3 Some Comments to Hydraulics of Free Overfall

For discharge measurements by low head devices of control structures, broad crested and long weirs are frequently used. As has been **briefly** indicated in 1-1-3, the hydraulic behaviours of flows over a broad crested weir are practically divided into three parts of entrance, channel and free overfall. In the entrance and free overfall portions, the curvilinear motion in fluid flows becomes predominant, while in the channel section between the entrance and the free overfall the hydrostatic pressure prevails. The weir serves as a control structure and it is related to the transitional characteristics of open channel flows. As the engineering approximation, the classical critical depth theory of gradually varied flows is used to estimate the discharge equation for head. Actually, the flow pattern is so complex in its behaviours that the complete description of basic physical characteristics in a mathematical form has not been established. Bearing in mind the above hydraulic behaviours, the brinking depth at a free overfall is often used to estimate the discharge rate, as Rouse[11)] did, in a form of

$$q = \sqrt{g}(h_b/0.715)^{3/2}. \tag{22}$$

Eq.(22) implies that the brinking depth is 0.715 times in depth of critical regime of parallel flow theory. Other famous relationships are formulas of A. Craya[12)] and C. Jaeger[13)], and especially M.R. Carstens and R.W. Carter[17)] obtained the graphical representation of flow characteristics of free overfall with the use of experimental data of T.H. Prentice[18)], Rouse[11)] and Fathy and Amin[14)]. These analyses have discussed the flow behaviours in connection with the critical depth of paralle flow in gradually varied flow theory. If the grade of broad crested weir is steep, the flow is shooting and consequently the weir can not serve as a control structure. On the other hand, for weirs of mild slope, the flow changes from tranquil to shooting at a point where is not definitely determined. In view of this matter, the analysis of Carstens and Carter is significant, as the hydraulic characteristics of free overfall are implicitely involved in their representation. All of results obtained theoretically and empirically by many hydraulic engineers describe that the brinking depth is commonly less than the critical depth of parallel flow of same rate in discharge. It means that the parallel flow theory in the gradually varied flow is invalid for analysis of hydraulic characteristics of free overfall. In this section, as the usual broad crested weir is equiped horizontally, so the simplest case of flow pattern will be considered by the present method of analysis. As indicated in Fig. 3-10, the flow of free overfall is assumed completely curvilinear, the total head from the bottom is

$$H_o = (q^2/2g)(R + h)^{-2}\{\log(1 + h/R)\}^{-2} + h. \qquad (23)$$

The radius of curvature at the terminal is apparently quite large whereas its value becomes finite in the immediate vicinity of downstream free nappe flow, so that the attention must be directed to the latter location. The pressure at the lower nappe just effluxed from the weir is atmospheric, and the pressure condition is expressible in terms of discharge characteristics, in a form of

$$h_b = 2R_b(gR_b/u_{sb}^2 - 1), \tag{24}$$

in which the subscript b indicates the values at the brink. With the use of total head, H_{ob}, Eq.(24) is solved for R_b,

$$R_b = (H_{ob} - h_b) + \sqrt{H_{ob}(H_{ob} - h_b)}. \tag{25}$$

Combining Eqs.(23) and (25), and introducing $q = (gh_c{}^3)^{1/2}$ and $H_{ob} = H_o = (3h_c/2)$, in which h_c is the critical depth in the parellel flow, as the first approximation, the ratio of h_b to h_c is expressed in the following,

$$\sqrt{3 - 2(h_b/h_c)}\{3 + \sqrt{9 - 6(h_b/h_c)}\}\log\{3/(3 - 2h_b/h_c)\}^{1/2} = 2, \tag{26}$$

and the resulting ratio of h_b/h_c is 0.673, or expressing in terms of total head, h_b/H_o is 0.450.

On the other hand, when the momentum procedure of approach is applied to the same flow at the free overfall, the momentum flux is expressible as, from Eq.(73) in 1-1-4,

$$M_o = (q^2/4gR^2)h(2R + h)^2/(R + h)^2\{\log(1 + h/R)\}^2 = H_oh - (h^2/2) + h\sqrt{H_o(H_o - h)}. \tag{27}$$

In the same manner as did for H_o, under the same assumption, the ratio of brinking depth to critical depth is a solution of

$$\sqrt{6}\{3 + \sqrt{9 - 6(h_b/h_c)}\}\{3 - 2(h_b/h_c) + \sqrt{9 - 6(h_b/h_c)}\}\log\{3/(3 - 2h_b/h_c)\}^{1/2} = 4(h_b/h_c)^{1/2}\{3 - (h_b/h_c) + \sqrt{9 - (h_b/h_c)}\}. \tag{28}$$

and it becomes

$$h_b/h_c = 0.690, \quad \text{and} \quad h_b/H_o = 0.460.$$

The distinctive discrepancy between both approaches is seen, and it will be caused by the irregular distribution of velocity and pressure known as the stability problem of Boussinesq. The definite solution may be only derived by more exact form of physical characteristics of free overfall flows verified by the experimental data. The above values are less than those of Rouse and Jaeger, while greater than that of Craya. Fig. 3-13 indicates the ratio of (h_b/h_c) measured by

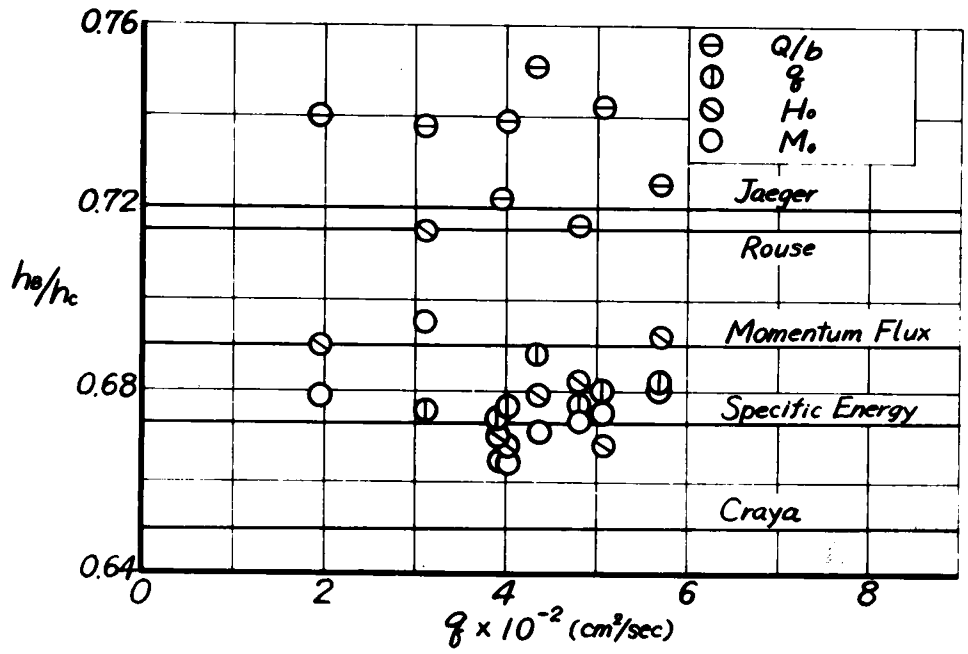

Fig. 3-13 Ratio of brinking depth to critical depth at free overfall

the long weir experiment conducted at the Hydraulics Laboratory, Kyoto University. In notations of the figure, the remark in terms of Q/b indicates the experimental value for the total discharge, q for the discharge per unit width obtained by the velocity profile at the brink, H_0 for the specific energy evaluated by pressure and velocity distributions at the same point, and M_0 for the momentum flux obtained by the same procedure. Experimental data are so scattered as to be practically not compared with theoretical values. Consequently, it is understood the discharge measurement of flows over a a free overfall by means of the measurement of brinking depth is so difficult.

With respect to discharge measurement by the free overfall, Rouse commented that the minimum energy was attained at the brink when the upstream tranquil flow passed in a long weir and channel. It is evident that the energy becomes minimum and at the same point the discharge becomes maximum when the basic equation of surface profiles involves a saddle point by the theorem of Bélanger-Böss. Consequently, the location of saddle point can not be derived if the x-wise and depth-wise distributions of velocity and pressure as seen in α and λ are not evaluated. The exact solution is still far away under the present knowledge of hydraulics of free oberfall, and in fact, if the foregoing description for H_0 is practically assumed as a first approximation, the brinking depth is 0.78 times of H_0 at the maximum discharge state. The hydraulics of broad crested weir, long wier, and free overfalls by means of the one dimensional procedure of approach is not certain, within the theoretical treatment and experimentation are at the present level. The treatment with the use of boundary layer theory are of recent trend as Ippen[5)] and J.W. Delleur[19)] did. However, no appreciable conclusions have not yet obtained through their analyses as well as the author's[20)].

---

1) Poleni, G., De motu aquae mixto, Padua, 1717.

2) Francis, J.B., Lowell Hydraulic Experiments, 1883.

3) Bazin, H.E., Expériences nouvelles sur l'écoulement en deversoir, Paris, 1898.

4) Rehbock, T., Discussion of E.W. Schoder and K.B. Turner's Precise Weir Measurements, Trans. ASCE, Vol. 93, 1929.

5) Ippen, A.T., Channel Transitions and Controls, Engineering Hydraulics, edited by H. Rouse, John Wiley, New York, 1950.

6) Paderi, F., Coefficient de debit et profil des barrages avec vitesse d'amenée non negligible, Proc. 6 th General Meeting, IAHR, The Hague, 1955.

7) Bleines, W., Berechnung des Überfalls über das scharfkantige Plattenwehr auf Grund des Extremalprinzips, Wasserwirtschaft, 43-43, 1953.

8) Mises, R.von, Berechnung von Ausfluss- und Überfallzahlen, Zeitschrift des VDI, 1917.

9) Kindsvater, C.E., and Carter, R.W., Discharge Characteristics of of Rectangular Thin-Plate Weirs, Jour, Hydraulics Division, Proc. ASCE, HY 6, Dec. 1957.

10) Böss, P., Berechnung der Abflussmengen und der Wasserspiegellage bei Abstürzen und Schwellen unter besonderer Berücksigung der dabei auftreten den Zusatzspannungen, Wasserkraft und Wasserwirtschsft, No. 2-3, 1929.

11) Rouse, H., Discharge Characteristics of the Free Overfall, Civil Engineering, Vol. 6, 1939.

12) Craya, A., Hauteur d'eau à l'extremite d'un long deversoir, La Houille Blanche, Mar. - April, 1948.

13) Jaeger, C., Hauteur d'eau à l'extremite d'un long deversoir, La Houille Blanche, Nov. - Dec., 1948.

14) Fathy, A., and Amin, M.S., Hydraulics of the Free Overfall, Proc. ASCE, Separate 564, Dec. 1954.

15) Schoder, E.W., and Turner, L.K., Precise Weir Measurement, Proc. ASCE, Vol. 53, Sept. 1927.

16) Kandaswamy, P.K., and Rouse, H., Characteristics of Flow over Terminal Weirs and Sills, Jour. Hydraulics Division, Proc. ASCE, HY 4, Aug. 1957.

17) Carstens, M.R., and Carter, R.W., Discussion of Hydraulics of the Free Overfall, by A. Fathy and M.S. Amin, Proc. ASCE, Separate 719, June 1955.

18) Prentice, T.H., Hydraulics of the Broad-Crested Weir, M.S. Thesis, Columbia University, 1935.

19) Delleur, J.W., The Boundary Layer Development on a Broad Crested Weir, Proc. 4 th Midwestern Conference, Flui. Mech., Purdue University, Lafayette, Indiana, 1955.

20) Iwasa, Y., Contributions of Boundary Layer Theory to Flows in Open Channels, Report, 1 st Symposium for Appl. Boundary Layer Theory in Hyd., Japan Academic Congress, April 1959.

5. Discharge Measurement by Control Flumes

3 - 5 - 1 Basic Consideration to Flumes

In the foregoing several chapters, the discussion of hydraulic functions of control structures is exclusively related to the so-called short structures which may cause the flow behaviours rapidly varied within a very short distance. The flow treated may be then classified as the rapidly varied flow. As the counterpart of rapidly varied flows, the gradually varied flows can be carried through the control structures. The typical example of control structures is various types of flumes.

Generally speaking, the flume is defined as the channel which consists of a strong contraction or expansion of the channel followed by an expansion or contraction, so that the flume is evidently a class of channel transitions. When the channel transitions are used for the discharge measurement of open channel flows, two water-level measurements are essentially needed, as the flow in such flumes will not change its flow regime. Such flumes are commonly called as the Venturi flume, which was invented by hydraulic engineers in the British colonies. Under certain conditions depending mainly on the discharge rate and the geometrical characteristics of channels in shape, grade and so on, the transition flows in flumes may involve the control section which is defined as the singular point of basic theoretical relationship of open channel flows in the control flume. The flume is then called as the channel control. It is readily understood that the construction of channel controls in the flume is the basic essentials of hydraulic design of flumes, for which the discharge is evaluated by a single water-level measurement. The Parshall flume[1], the flumes of Sir Inglis and of de Marchi[2] are typical examples of such control flumes.

As described in the foregoing chapter, the weir is also the control structure. The choice between weirs and flumes as measurement devices for discharge depends mainly on the hydraulic functions of each type. The weir is simpler and presumably, less expensive, but it consumes a large amount of head.

On the other hand, the flume is favourable for flat areas as it is a typical low head device, and also it is less influenced by sediment deposits than those of a weir. However, the low head control structure is readily influenced by the downstream condition, so that particular attention in hydraulic design of flumes as control structures and not as transition structures must be accented. The other important consideration to hydraulic design of flumes is the degree of submergence. The flow which will be measured by the flume changes tranquil to shooting and again the shooting flow will be changed to the tranquil one for preventing the structure from the unfavourable local scour downstream. The downstream water elevation is thus increased by some suitable devices. If the submergence is too large, the influence of downstream flow condition may travel over the flume and sometimes far upstream from the flume, resulted in no availability for measuring devices.

A large variety of control flumes as critical flow meters have been invented by many hydraulic engineers, and famous are those of Parshall, Inglis, de Marchi and Crump[3] The hydraulic behaviours of flows in these flumes are similar and classified as the transitional characteristics of gradually varied flows, and the head-discharge relationship as the most important relationship of discharge measurement will be also established by the theoretical calculation. The basic surface profile equation in the flume is evidently non-linear in terms of channel geometries and roughness as well as discharge.

The head-discharge relationship is not explicitely expressed in a simple form like solutions of linear equation. For examples, Parshall proposed the discharge equation of Parshall flume in a form of

$$Q = 4Wh^{1.522W^{0.026}} \tag{1}$$

in which W and h are the throat width of the Parshall flume and upstream depth at the gauging well. Similar examples are frequently observed in many hydraulic literatures. Consequently, the head-discharge relationship of control flumes can not be generalized owing to non-linearity in the basic flow behaviour. Only possible solution for discharge measurement is substantially provided by each flume with a definite dimension of constant channel geometry, and the universal type of discharge coefficient for flumes can never established.

The hydraulic design procedure is thus in many ways. The most significant design in the light of the knowledge in theoretical hydraulics with systematic experimental verification is not existed. The hydraulic standardization of flume dimensions is further problems to establish for various hydraulic works included in the gravity projects.

The experimental verification of transitional characteristics of gradually varied flows in control structures will be treated in the first place. The test flumes are a divergent flume and a common flume with channel contraction and expansion. In the first flume, the channel geometry changes continuously, whereas the channel geometry of the second flume is not continuous in the part of expansion and contraction. Nevertheless, the theoretical calculation of the transitional characteristics by means of the foregoing parts of I and II will indicate a very close agreement with the experimental data. Next will be discussed some comments of Parshall flumes and the hydraulic requirement of flumes to measure the discharge by a single water -level measurement.

3 - 5 - 2 Experimental Verification of Transitional Characteristics of Gradually Varied Flows in Flumes

General transitional characteristics of gradually varied flows have been discussed in 2-4. The flow in flumes is a typical example of gradually varied flows with transitional behaviours. The experimental verification of transitional characteristics of the flow will be concerned in this section.

The first run of experimental program was conducted in the divergent flume, illustrated in Fig. 3-14, at the Hydraulics Laboratory of Kyoto University. The flume is made with lucite and the channel grade is set at 1/500. The dimension of flume is also indicated in the figure. When the water is carried through the flume, the possible surface profiles are evaluated by means of the foregoing description. For a given discharge, the surface prifile equation is

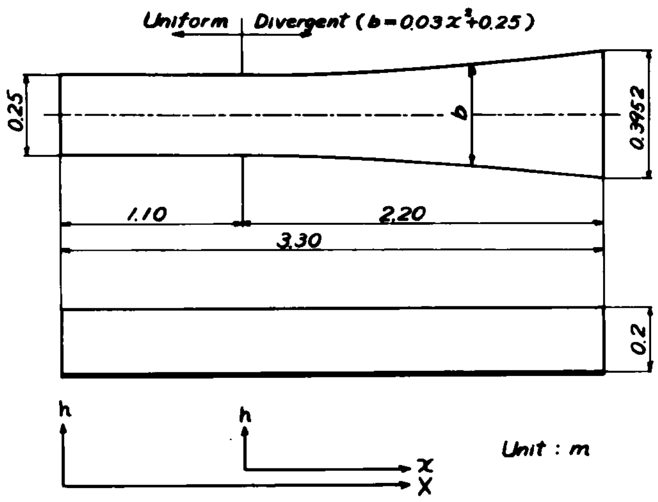

Fig. 3-14 Dimensions of divergent flume

$$(dh/dx) = f_1(h, x)/f_2(h, x) \tag{2}$$

where

$$f_1(h, x) = 0.0196 - 0.0007938(b + 2h)^{4/3}(bh)^{-10/3} + (Q^2/b^2h^2)(db/dx),$$

$$f_2(h, x) = 9.8 - (Q^2/b^2h^3),$$

and the Manning formula is applied. The roughness of flume is assumed 0 009 (m-sec). Normal and critical depths curves are calculated for given discharge, which are illustrated in Fig. 3-15. Intersecting points of two curves are evidently singular points and easily proved as saddle point for all values of discharge. All surface profiles involving the transition curve for a particular discharge in this flume are numerically calculated by integrating Eq.(2) with the use of the foregoing description. Fig. 3-16 indicates experimentally observed curves of surface profiles under various downstream conditions for Q = 5 l/sec, with calculated curves under the same conditions. When the downstream water elevation is high, the flow is submerged and the flume is classified as channel transition. As the downstream influences can travel upstream through the flume, so the estimation of discharge requires a two water-level measurement. When the downstream water level is lowered, the flume is functionalized by controlling of flow regime. In this run, the water level for dividing

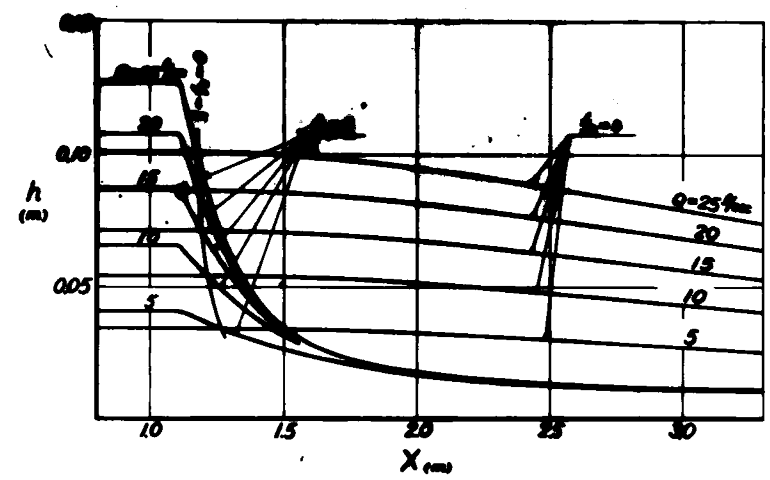

Fig. 3-15 Normal and critical depths curves in first run of experiments

the flume in its performance into a channel transition and a control is 5.09 cm at the downstream end. The surface profiles are traced

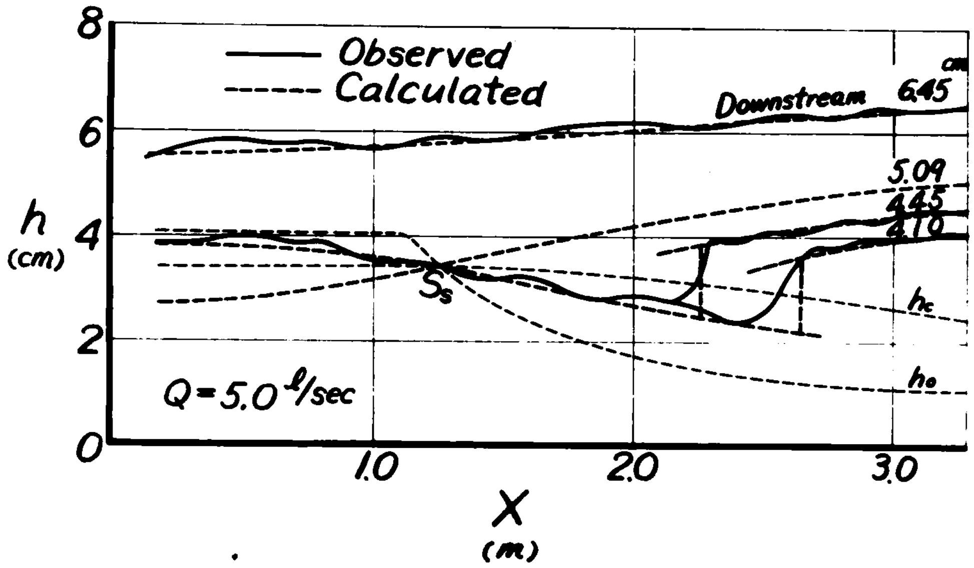

Fig. 3-16 Surface profiles of flows in divergent channels

by connecting two curves of transition profiles through the saddle point and of downstream flow curves at a point where both depths becomes conjugate as has been described. Of special evidence is a very close agreement between theoretical curves and experimental data and the conclusion of the validity of foregoing description of transitional characteristics will be obtained.

The second run of experiment was made in the wooden flume coated with white paint, which is also illustrated in Fig. 3-19. The flume has a throat in the middle portion. The ratio of width change with respect to distance is discontinoous at x = 1.2, 1.8, 2.1 and 2.7 m, so that the normal depth curve has two values at discontinuous point, while the critical depth curve is of single value function throughout the whole flume. The definite locations of singular point can not be evaluated, as the basic equation is not continuous. In practical calculation of surface profiles in uniform channels with variable slopes, the discontinuous points of two curves of normal and critical depths may be possible locations of singular points, and it will be assumed that a saddle point and a nodal or focal point are produced at 2.1 and 2.7 m, respectively, remembering that the general classification of transitional characteristics of gradually varied flows. Consequently, the computation of surface profiles will approximately start from the point where two curves of normal and critical depths intersect together at 2.1 m. Fig. 4-18 illustrates the surface

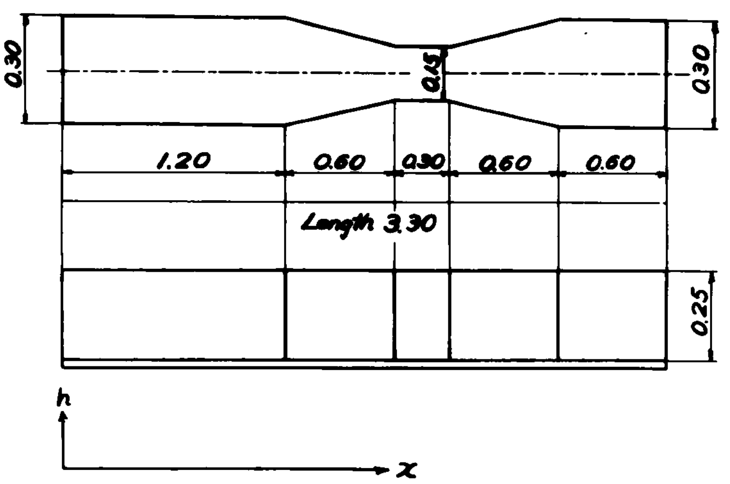

Fig. 3-17 Dimensions of flume used in second run of experiments

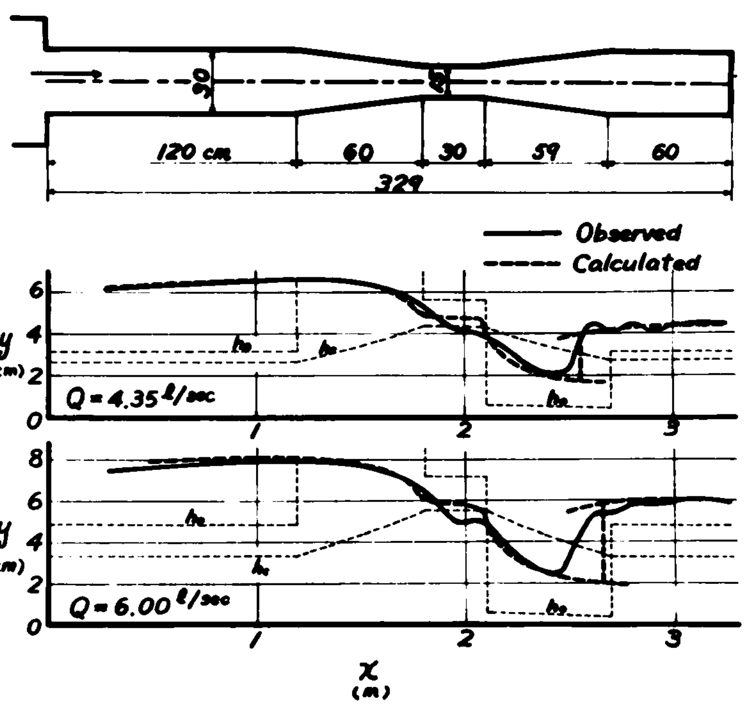

Fig. 3-18 Surface profiles of water in various cases of discharges in channel transitions and controls

profiles of water experimentally obtained for Q = 4.35 and 6.00 l/sec, respectively. In these experimental runs, the downstream water stages are regulated at 4.40 and 6.10 cm by the controlling gate, so that the hydraulic jumps are occured in the expanding part of the channel. Dotted lines indicate calculated curves of surface profile for same rates of discharge under the assumption of n = 0.009 (m-sec). It is evidently seen both curves are closely agreed, and the discharge metering by a single water-level measurement in the flume is made possible with sufficient accuracy.

As seen in two experimental runs made in different types of open flumes, the verification of transitional characteristics of gradually varied flow is established. Even in flumes constructed discontinuously in wall alignment, the surface profiles of water are satisfactorily estimated except in the very immediate vicinity of singular point.

3 - 5 - 3 Hydraulic Functions of Flumes for Discharge Measurement

As seen in the preceding section, the flow behaviour in flumes with channel contraction and expansion can be described by the trnasitional characteristics of gradually varied flows, and consequently it refers the discharge may be measured by flumes which are classified as the control structure. Various types of flumes like Parshall, Inglis, de Marchi and others are examples of those structures for discharge metering.

For practical uses, the head-discharge relationship for a particular type of flume is described in various forms, among which the power law as a semi-rational relation is very popular. The formula of Parshall for the Parshall flume, which has already indicated, is an example. The head-discharge relationship of the flume for the second run of experiments described in the preceding section, is obtained by

$$q = 0.296h^{1.548} \quad (\text{m-sec}),$$

for q = 5 - 20 l/sec, when the water depth is measured at x = 1.4 m.

However, these formulas involve no dynamic evidence for the basic flow in flume when deducting as an explicite form and generally the complete mathematical description for the head-discharge relationship can never be obtained owing to the non-linearity in the flow behaviours. Only the possible way for the complete formulation is to make the tabulation with the use of a large number of experimental studies and analytical treatments. In this meaning, the tabulation of Parshall for various sizes of his flume is desirable. In view of this point, the complete tabulation for discharge characteristics in control flumes is possible by the tremendous amount of labours, as the flow is described by the transitional characteristics of gradually varied flows.

All types of flumes involving the Parshall and other flumes, however, consist of several portions of channel contraction and expansion with discontinuous changes in alignment, so that the precise estimation of discharge characteristics can not be made by the theoretical analysis. Actually, in the Parshall flume, the locations of saddle points as transitional point from tranquil to shooting are not definitely evaluated, and therefore the theoretical estimation for this type of flume will only give a rough relationship for head and discharge as the engineering use. Although hydraulic engineers often propose their own empirical formulas for the head-discharge principle through their experiments in a particular type of flume, it is evident that the formulas proposed are available for the flume used in the experiment and the establishment of the universal law for discharge measurement can never be produced by the empirical procedure.

For practical purpose of discharge metering, the favourable design criteria for flumes as control devices, which will be verified by the analytical and experimental studies, are expressible as follows.

(1) Channel geometry and grade must be selected so that the location of saddle point in the flume is completely evaluated by the theoretical analysis, and therefore the channel alignment and grade are usually designed to be gradually varied.

(2) Channel geometry and grade also must be chosen so that the saddle point is always occured in the flume for the whole range of design discharges.

(3) Considerations to pressure requirement is not generally needed.

(4) Additional attentions have been already described in 3-5-1.

---

1) Parshall, Measuring Flume, Fort Collins, Colo., 1921.

2) de Marchi, and Contessini, Dispositivi per la misura della portata dei canali con minima perdita di quota, Energia Elet., Jan. 1936, Mar. 1937.

3) Citrini, Misuratori a risalto, Politecnico di Milano, 1939.

IV. HYDRAULIC CONTRIBUTIONS TO DESIGN PROBLEMS IN CONVEYANCE AND CONTROL STRUCTURES

1. Hydraulic Requirements for Design of Conveyance Structures

The irrigation projects are agricultural establishments where controlled water supplies are applied to lands in order to produce crops. The power development projects are real contributions to the economic projects of a nation, and the projects of stream regulation and flood control are closely related to the promotion of welfare in human welfare and of social and economic circumstances of a nation, directly or indirectly, in various ways. All of these hydraulic projects called commonly gravity projects are usually of very large scale and in some certain cases it is nationwide. The general scheme of design in hydraulic projects is therefore to try successively varying combinations of planning elements until the most economical combination of objective in hydraulic works is found. Projects elements in a stage of planning development are not only hydraulic structures but also topographical and geological conditions in the planning area, economic factors, analyses of hydrological and hydrographical data and so on. Topographical and geological conditions have greater influences on the selection of location and type of conveyance structures than any other factors. The class and amount of excavation, possibility of seepage and piping, values of excavated material for other purposes, possibility of scour and consequent need for lining and others must be carefully considered. As the hydraulic projects are subjected to storage and conveyance of water from the engineering standpoint, so comprehensive hydrological and hydrographical studies give the general stream flow characteristics involving daily stream flow records, maximum discharge of flood and others. The outlet and diversion capacity requirements can be thus determined.

After the determination of basic hydrological elements is finished, hydraulic design of various structures of conduits as the final stage of planning works in the project will be made. There is, however, no general principles for layout and development of hydraulic projects as combined features of basic various requirements, being subject to so many variable conditions at a particular site. The exercise of great care and mature judgment in design to insure economical, practical and safe conditions are required.

Many different types of structures are needed on hydraulic projects. These may be, however, divided in two general classifications, (1) structures required in obtaining water at sources of supply, and (2) structures required along conveyance routes from sources of supply to the demanding area. Types of hydraulic structures that must be provided at sources of supply are dams and pumping plants. Storage and diversion dams must be accompanied with conveyance and control structures like adequate spillways, outlets and control gates. Intakes, usually located at or upstream from diversion dams are equiped with desilting basins or other special appurtenant structures as well as ordinary conveyance and control structures. Structures of various types and sizes are required along the conduits that carry water from sources of supply to demanding portions. Such hydraulic structures in conveyance system are gate-control works, spillways and wasteway in large scales and drops, chutes, culverts and other minor features in small scales, any or all of which may be referred to as canal structures.

Different canal structures required on a typical gravity project may be classified according to purposes as conveyance structures, control or regulating structures, protective structures and miscellaneous structures. Conveyance and control structures include the various canal structures that might be installed on a particular hydraulic project in order to maintain and control discharges and and water elevations as well as other flow characteristics through all parts of project area. The following outline given by I.E. Houk[1] lists the principal structures that may be included in each classification from the point view of practical engineering purposes.

(1) Conveyance structures

- i Inverted siphons under obstructions or across drainage courses
- ii Flumes across drainage courses or other depressions
- iii Transitions at changes in cross section
- iv Drops to lower water surfaces in open conduits
- v Chutes down relatively short steep slopes
- vi Culverts to carry flows under highways or railways

(2) Control Structures

- i Division works at heads of branch canals
- ii Turnouts to small laterals or ditches
- iii Checks to raise water surfaces in open conduits
- iv Wasteways to discharge excess canal flows

(3) Protective structures

- i Spillways to prevent overtopping of canal banks
- ii Overchutes to carry flood runoff over open conduits
- iii Culverts to carry drainage water under canals
- iv Drainage inlets to canal sections

(4) Miscellaneous structures

- i Sand and gravel traps
- ii Desilting basins
- iii Bridge across open conduits

These classification are, to some extent, arbitrary, and the features planned for different purposes frequently are incorporated in the same structures. In fact, hydraulic structures classified as the protective structures for certain purposes of protection from damage may be usually built as elements of conveyance or control structures.

Limiting the problem only to the open channel, the general procedure in hydraulic design is evidently one of water surface determination. The required height of side wall must be determined by adding the necessary freeboard to the carefully estimated water depth. The cross section and channel grade of the final design of canal structures for particular design discharges by a comparison of the combined excavation and other cost for several tentative layouts estimated by the flow characteristics resulted from the determination of surface profiles.must ensure the safety-carriage of water and the prevention of unfavourable local damages. As has been discussed in the foregoing parts, hydraulically speaking, the determination of flow characteristics involving routing of surface profiles of water is essentially resulted from the transitional characteristics of open channel flows: flows in channel transitions and those in channel controls. At a control section in channel controls all hydraulic characteristics of flows are uniquely determined for given rates of discharge, and this section becomes a starting point to proceed the computation of surface profiles. The control structure is defined as the class of structure in which a control section is involved for given rates of discharge in this study. Therefore, being different from the classification of control structures by Houk who defined the classification of hydraulic structures from the standpoint of practical requirements for engineering purposes, various types of canal structures become control structures under particular conditions. On the other hand, in channel transitions usually defined as local changes in channel geometry, grade and boundary materials which produce a variation from one uniform to another, the surface profiles must be proceeded up- or downstream depending on the up- or downstream control conditions. Channel transitions are involved in nearly all canal structures, and even in artificial watercourses in gravity projects. One of main purposes of canal structures is to carry the water, so that in the present study, the conveyance structure is defined as open channels not involving channel controls. Uniform channels and transition channels are included in this classification.

Aside from the present definition of conveyance and control structures, basic requirements and hydraulic characteristics of various types of canal structures will be briefly considered.

At entrances to channels and at changes in cross section and bed grade, the structure which conducts the water from the upstream section to the new section is called a transition. Hydraulically speaking, the transition is classified into two transitions for tranquil and shooting flows. The transition for tranquil flows usually is required at the inlet and outlet of channels, flumes, siphons, pipes, and tunnels along main conveyance route. The design principle of structure is prevention of disturbances in flow, the minimum losses of energy where velocities are increased, and the recovery of velocity head where velocities are decreased consistent with economy of construction. Transitions also are subdivided into channel expansion, channel contraction, positive and negative steps. The shape of transitions is made bellmouthed, curved and warped. On the other hand, in the transition for shooting flows, the disturbance lines known as shock front are developed by the boundary deflection, as the small boundary deflection can not influence surface conditions upstream in the supercritical flow. The flow characteristics rapidly change through the shock line, with results of formation of considerable height of shock waves. Careful considerations to estimate the height of side wall are needed. For some particular or all discharges, the transition may become a channel control, in which the hydraulic characteristics are uniquely determined. Whether the transition may be a control or not is evaluated by the procedure described in 1-2-4.

Drops are designed to accomplish lowering of grade and water surfaces in relatively short distances without damage to downstream connecting conduits. In small channels and ditches, the drop structures are often more suitable than other types of lowering structures in grade and head. They are also used for purposes of desilting works and discharge measurement. Various types of drop structures are resulted from the appurtenant structures included in the drop structure. The drop structure may be considered as the terminal of a channel where the flow suddenly changes, so that the flow characteristics over a drop structure are obtained by the hydraulics of free overfall described in 3-4-3.

Chutes which are similar to drops in their purposes are used for the release of surplus water in excess of the demanding capacity. Although no definite distinction can be made between drops and chutes, the latter is generally longer and larger than the former. A chute is of type of steep open channel and may be straight or curved, with the side parallel, convergent or divergent in plan, so that it involves in common several transitions and control sections. The design of a chute is largely fixed by the topographical conditions, and for tentatively selected site, hydraulic design of chutes can be made as the determination of water surface profiles. Particularly significant in hydraulic behaviour of flows in chutes is that the flow is extremely shooting. In straight and long chutes, the flow is air-entrained, the bulk discharge is increased by the volume of entrained air into the flow. When the chute is curved, the formation of strong shock wave is resulted. Therefore, the determination of proper freeboards along chute channels must be based on careful hydraulic considerations of surface profiles of water, probable amount of air-entrainment and height of successive shock waves. In very long chutes, the formation of roll waves resulted from the hydraulic instability of shooting flows and the boundary resistance is frequently observed when the released discharge is small in amount. The height of frontal waves which are similar to those of the breaking surge is several times higher than that of normal flow for the same rate of discharge. Ample height of freeboards is also required to secure the safety-carriage of released water and the prevention of overtopping from the channel wall. Stilling basin at the end of chute is usually built to dissipate the vast kinetic energy by means of various types of energy dissipators.

Division works usually consist of gate structure at the head of each branch canal, so that they are control structures and the flow in each branch can be regulated as required. From the standpoint of hydraulics of open channel flows, the flow characteristics over a division structure are essentially determined by regulating gates, when the gate is operated at partial opening. On the other hand, when the gate is fully opened, the crest of division structures becomes generally a control section, and therefore, the discharge characteristics are similar to those of uncontrolled crest of overflow spillways and can be evaluated by the theory of curved flow described in the foregoing part.

Checks are installed in canals so that the surface elevation of water can be raised as required during canal operation. They often permit delivery of water to higher area that otherwise could not be irrigated. Most checks are short structures with water surface controls like various types of gates and flashboards across the channel and transition sections up- and downstream from the control. Evidently, the check is a control structure to regulate the water stage. In the computation procedure of surface profiles of water for canal design, however, it gives the initial value in calculation of basic equation, and in up- and downstream transitions and controls resulted from changes in channel section and bottom grade, various types of surface profiles involving common backwater curves, smooth transition curve from tranquil to shooting flows and hydraulic jump may be produced by the check.

Wasteways are built to divert canal flows into natural channels or other suitable outlets as required during canal operations. Some wasteways are designed for multiple purposes of sluicing sediment and releasing surplus water. The design capacity of wasteway must be equal to maximum flows that may occur in the canal. A typical waste-way usually includes a gate structure near the side of the canal. The sill of gate at the structure is generally placed at about the same elevation as the canal bed, and is, in some particular cases, lower than the canal bed where the wasteway is to be used for both sluicing and diversion. The gate of wasteway can regulate the discharge and the water elevation and the downstream surface profiles are estimated by the procedure described in 2-4. The downstream channel is commonly of very mild slope, so that the flow becomes submerged.

Spillways provide controlled discharge in excess of reservoir and canal capacity. By the definition of Houk, spillways are in the class of protective structure, as its use is to prevent rises in water surfaces that might damage the banks of the dams and cause serious interruption in service. A typical example of spillways' function to discharge characteristics has been treated in 3-3.

As apparently understood in the foregoing description of various types of canal structures, hydraulic designs of conveyance and control structures as well as canal itself are to estimate definitely the flow characteristics of stream flow throughout the whole part of structures and canal. The hydraulic analysis of open channel flows is substantially furnished with the one dimensional procedures of analysis derived by the energy and momentum conservation laws and consequently the design problem of hydraulic structures is attributed to the determination of surface profiles for particular discharges obtained by the hydrological and hydrographical studies of the project, or sometimes, given by the policy-maker. The hydraulics of steady flows gives thus the basic important features to the design procedure, as nearly all canals and other hydraulic structures are operated for a definite range of discharges. The flood is a typical example of unsteady flow phenomena, which are a counterpart of the hydraulics of steady flows exclusively described in this study. Nevertheless, the flow behaviours of flood flows near maximum stage and discharge are often assumed quasi-steady as the engineering approximation, and the design problem to determine the bank height also is analyzed as the steady flow problem. Furthermore, the theory of the propagation of surge front is applied to estimate the height of channel walls in power and navigation canals.

Open channels as devices to carry waters for various purposes of irrigation, power developemt , flood controls, and navigation are then divided into conveyance canals of mild and steep slopes and control structures, from the point view of hydraulic characretistics of flow in channels. Many other features in hydraulics urge special considerations of basic hydraulic characteristics to furnish the complete determination of flow characteristics in design purposes. In this part, some contributions to design procedures of various types of hydraulic channels will be presented to improve the further development in hydraulic works as well as hydraulic research projects of open channel flows. In the first place, the hydraulic treatment for design of canal structures involving channels and channel transitions is concerned, and the first section of the chapter treats with the basic requirement and the hydraulic characteristics of mild slope channels, which are in common for various purposes of irrigation, navigation and other hydraulic projects in flat areas. The second section deals with the hydraulic requirements for designs of steep slope channels, which are observed in power development and flood control projects. The flow in steep channels is highly shooting and many outstanding behaviours are accompanied not seen in tranquil flows. The basic requirements for channel transitions will be described in the last section. The next chapter concerns with the basic requirements of control structures when the design procedure is made.

4 - 1 - 1 Conveyance Structures of Mild Slopes

In general, the channels are planned to carry required maximum discharge for the project purposes. Additional studies are needed to obtain the most economical and suitable types of channels and design details, which involve the geometric dimensions necessary to construction. Construction costs and future operation and maintenance costs also must be included in design details.

Mild channels are basically characterized by less value in slope than the critical slope. The normal flow in mild channels is tranquil with low kinetic energy. Cross sections in channels must be planned so that the available channel grades can maintain required discharge at mean velocities between the lower and upper permissible limits given by various purposes, though the channel grades are primarily determined by the topographical condition. Some canals, in which the velocity of flow is not exceeded over a limiting value of critical tractive forces, are not generally required by channel lining, whereas at particular sections near transitions and controls included in channels, the lining is needed. The cross section is usually rectangular and trapezoidal in artificial channels and arbitrary but similar to rectangular in natural watercourses.

The hydraulic design of a straight and mild channels with uniform section is the simplest problem of canal construction in determination of surface profiles. The required height of channel walls and banks is obtained by adding the necessary freeboard to the water depth. The determination of the surface profiles of water is essentially based on the Bernoulli theorem of energy conservation described by Eqs.(68) and (69) in 1-1-4 and the momentum conservation law of Eqs.(74) and (75) in 1-1-4. The flow in such channels is also characterized by the term of gradually varied, and the hydrostatic pressure prevails in the flow and correction factor due to irregular distribution of velocity will be practically assumed a constant value of 1.00 - 1.05. The calculation procedure is proceeded in the direction of upstream as the flow is downstream control. Many fruitful studies described in 2-2-1 are available for the determination of surface profiles, and especially the chart and the table are most effective for this purpose. When a control gate is located in the channel, the hydraulic jump may be occured downstream in forms of undular, roller and submerged depending on the downstream flow conditions. The calculation is started from this point in the structure. The classification of types of hydraulic jump is described in 2-3. Special considerations to determine the wall height must be put on the flow in which the undular jump may be occured as described in 2-3-3. If the bottom grades are changed at several points in channels, the channels is still mild over the whole part as the terminology "mild slope" is assumed, so that the flow is essentially downstream control and no locations of possible control section are resulted. When the drop and chute structures are followed at the downstream end of mild slope channel, the water elevation at the free overfall is decreased by about 0.7 time amount of critical depth of parallel stream flows for the same discharge, as described in 3-4-3. Furthermore, the flow at the free overfall is not critical in the classical theory of hydraulics, and therefore the computation can not be proceeded. However, it is seen from the experimental data[2)] that the surface profiles near the free overfall is expressed by a single curve of dimensionless distance from the free overfall and the critical depth of flow in terms of the parallel flow theory is obtainéd at a point where the hydrostatic pressure prevails. With the use of this description, the surface profiles of water can be practically determined by integrating the basic equation of gradually varied flows from the upstream section in the immediate vicinity of critical depth point. The usual procedure of computation in this case is approximately started from the downstream end of free overfall. It is obtained by many hydraulic engineers that the computed surface profiles of water indicate a close agreement with the actual water elevation except the curve near the free overfall. The flow in the immediate vicinity of free overfall includes no series of undulations resulted from the vertical acceleration as described in 2-2-1, and no additional consideration to determine the freeboard of channels is required. On the other hand, in the upstream reach, the undular form of surface profiles is often observed and therefore the supplemental analysis will be needed.

When the sudden increase of discharge is released in the mild channels like the outlet of power canals and the wasteway, the surge front will be progressed. As the height of front waves of surges and translation waves is not easily damped out when the wave is travelling downstream through the channel, as described in 1-3-4, the careful treatment for freeboard of channels must be considered in power canals and other types of canals in which the translation waves and surges are travelled.

One of the most complicated and difficult problems of hydraulics in which little progress has been made is the flow through the curved channels. Although the available procedure to determine the surface profiles of water in curved channels has not been obtained, the method of approach of USBR that made the procedure for hydraulic design of curved spillway may be used as a first approximation based on the theory of concentric flow locally parallel to the channel wall. In mild channels of curved boundary, the velocity is low, and the resulting superelevation at the outside of channel is also of small value. More significant features in the flow than the surface configuration are that the maximum velocity is located at other point from the center and the spiral motion and secondary currents are induced in the stream, as has been indicated experimentally by many scientists and engineers for many years. The universal treatment of design procedure for curved channels has not yet been established because of great complexity in basic flow patterns, so that the model study for tentative several plannings of the curved channels is largely recommended for obtaining more effective and practical solutions of this problem. The problem remains unsolved until the transitional chatacteristics of two dimensional flows by means of the boundary layer theory and other treatments are completely established. The hydraulic design of unlined canals must be made with the model study. More contributions of flow patterns in curved channels obtained by many experimental studies will be surely resulted in the further success in theoretical hydraulics.

4 - 1 - 2 Conveyance Structures of Steep Slopes

Steep channels are the class of channels in which the flow is generally shooting, when no control structures are in the channel, though the definition of steep channels is different from the usual one in the theory of open channel flows. The steep channels of uniform section are reflected by the characteristics of shooting flows, which are commonly emphasized by the fact that large variation in total head and momentum flux is resulted little in the value of the depth. Many outstanding features in the shooting flow not found in the tranquil flow are pronounced in steep channels of uniform and non-uniform sections. This section concerns with the basic requirements for design of steep channels with constant section.

The hydraulic design of steep channels of uniform section is the simplest case for design problems of chutes, spillways and other similar conveyance structures of steep grade as well as for those of mild channels. The design requirement is attributed to the determination of surface profiles of water for particular design discharges. In the same manner as did for mild channels, the determination of surface profiles for the definite discharge is made by means of the Bernoulli theorem and the momentum equation for straight channels of constant grade. The classification of surface profiles resulted from other control structures are feasibly seen in many hydraulic literatures. As described in the above statement, little variation in head are caused by the change of head and momentum flux, so that the estimation of surface profiles is more likely than that in tranquil branches. No formation of surface undulation is resulted in the shooting flow, if the magnitude of vertical acceleration is appreciable. The quite different characteristics of shooting flows in steep channels are the formation of roll waves and the air-entrainment into the flow.

The formation of roll waves in steep channels yields the continual changes in water elevation and sometimes the flow will overtop from the channel walls designed under the condition of steady uniform flow of design discharge. Consequently, one of basic requirements for hydraulic design of steep channels is to prevent the overtopping of water flow from the channel wall, considering the hydraulic characteristics of roll waves. As shortly described in 1-2-5, the roll waves frequently called as the intermittent surges or the slug water are essentially resulted from the hydraulic instability of shooting flows. The criterion for initial instability of flows of Vedernikov, or the revised formula presented by Iwagaki and the author are indicated in 1-2-3. When the channel is of rectangular shape and the flow is practically assumed of uniform type in velocity distribution, the instability will be initiated at the value of Froude number which is

for Chézy flows, $F_0 = 2$,

and for Manning flows, $F_0 = 1.5$.

It is therefore seen the wide and steep channels may be inherently involved by the possibility of roll wave formation. Once the roll wave is generated, so that it continually increases to become its final wave patterns, as precisely discussed in the literature of Iwagaki and the author[3]. The maximum water depth is considerably increased at the wave front compared with the depth of normal flow for the same discharge. The ratio of roll wave height to normal flow depth increases with the increase of Froude number. For examples, the theoretical analysis based on the shock wave theory used by Thomas, Dressler, Iwagaki and the author indicates the ratio becomes three at the value of 5 in Froude number and five when the Froude number becomes 9, so that the tremendous increase of water depth at the wave front is observed. The careful treatment for hydraulic design of channel wall and freeboard of steep channels must be considered.

When the stream flows down from the overflow spillway and steep chute, the air-entrainment into the flow is commonly observed. The air-entrainment process has not been theoretically established because of its great complicated phenomenon of air-water mixutre, despite of much endeavours by many hydraulic engineers since E.A. Lane[4]. The process, however, may be assumed to occur when the high degree of turbulence of the flow exceeds a certain minimum beyond which a water particle at the air-water boundary possesses some finite magnitude of vertical component, the magnitude of velocity of particle in air is large to catch the air and the flow condition necessary to maintain the continuation and growth of air-entrainment is remained. Although there is a lack of theoretical results of air-entrainment, the engineering need for hydraulic design of steep chutes, discharge carriers of overflow spillways and energy dissipators enforces to the experimental study. Many papers concerned with the entrainment of air and design problem of freeboard are presented by a large number of engineers.

The widely famous study of this problem is to concern with the initiation condition of air-entrainment. As Lane first suggested, the initiation of air-entrainment is closely related to the turbulent boundary layer growth in the flow. The Prandtl-Schlichting equation of turbulent layer on a flat plate is familiar to hydraulic engineers and exclusively used as the index of growth parameter. The turbulent layer growth of open channel flows as confined flows was first studied by G. Halbronn[5] in 1952 as already cited. In the same year, A. Craya and J.W. Delleur[6] also investigated the same process of flows on a slightly sloping channel. Especially, the former investigator made much contributions to the turbulent layer growth in connection with the actual problem of hydraulic design. In 1953, W.J. Bauer[7)] obtained many fruitful experimental data of turbulent layer growth in open channel flows with systematic experimentations. As has been discussed in 1-1-5, the initiation point of air-entrainment may be treated by estimating the limiting condition of boundary layer growth, if the boundary layer may be closely related to the process of air-entrainment. In actual channels, the flume is made with concrete, so that the critical condition for the initiation of air-entrainment expressed by Eq.(108) in 1-1-5 must be modified by the increase of channel roughness. For this purpose, the knowledge of velocity distribution of flows over a concrete flume is also required. As a first approximation, the velocity profile in the turbulent layer near the concrete bed is assumed to be of 7 th law, so that the skin friction law will be expressible as, at the critical point,

$$(\tau_c/\rho) = (C_f/2)u_c^{\,2} = \lambda_{oc}(\nu/u_c h_c)^{1/4}, \qquad (1)$$

in which λ_{oc} is the coefficient of resistance of concrete. At the same point, the Manning law for mean value of velocity in the flow is

$$(\tau_c/\rho) = gq^2n^2h_c^{\,-7/3}, \qquad (2)$$

where n is the Manning roughness for concrete surface and assumed 0.013 (m-sec). With the aid of Eq.(107) in 1-1-5 for turbulent layer growth in accelerative flows and the constancy of discharge, the relationship between the Manning roughness n and λ_{oc} will be obtained in an approximate form. It is not constant, but for a wide range of discharge, n is linearly increasing with the increase of λ_{oc}, and the value of 0.013 will be estimated as 0.0770 for λ_{oc}. The critical condition, therefore, is established. Fig. 4-1 indicates the distance

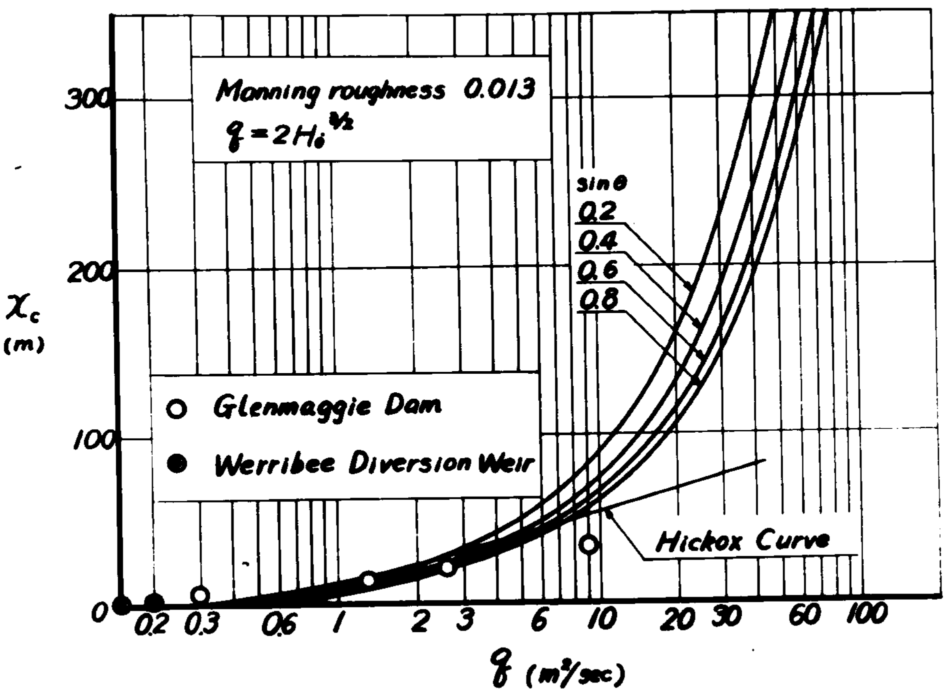

Fig. 4-1 Relationship between initial point of air-entrainment and discharge on steep slopes

of this point for various values of discharge per unit width, under the assumption of $q = 2H_0^{3/2}$ for the discharge characteristics of overflow apillway crest. In the same figure, the empirical curve of Hickox for the initial point of air-entrainment obtained by field observations and experimental data of V. Michel and M. Lovely[8] at the Glenmaggie Dam and the Werribee Diversion Weir are presented. It is rather surprising that the rough estimation of initial point for air-entrainment of flow over spillways is closely a representative of actual observations, and consequently, the concept that the turbulent layer growth in open channel flows over a steep chute is related to the process of air-entrainment will be supported.

After establishing the air-water mixture, the apparent depth will be increased by the intrusion of air bubbles. D.B. Gumensky[9] reported the following design procedure for channel walls of spillways, chutes and other steep slopes, as the design criteria used by the U.S. Office of Engineers at Sacramento, California. For spillway designs, the first approximate calculation must be made to estimate the actual velocity and water depth by the use of the Manning formula in which n is assumed to be 0.008. The estimation of bulking height due to the air-entrainment is made by the second calculation in which the Manning formula is again used. In this case, n is assumed 0.014. The result of apparent water depth involving the bulking is considered approximately valid. The ratio of air to solid water is given by

$$m = 0.0000472(u_m^2/h), \quad (\text{m-sec}) \tag{3}$$

with the use of L.S. Hall's[10)] experimental results. As a conclusive statement for the recommended design procedure of channel walls, Gumensky obtained that the height of channel wall must be designed for 100 % air-water mixture, adding at least 1.5 m of freeboard in height.

The problem of the air-entrainment is still not evident, and especially the dynamics of air-water mixture has not been developed even in a simple flow pattern. The design procedure for hydraulic design of channel walls must be largely based on the empirical relationships described in the above explanation.

4 - 1 - 3 Hydraulic Design of Channel Transitions

Natural channels consist generally of a series of successive channel transitions which are defined as local changes in channel geometry, roughness and bottom grade. Even in man-made watercourses of constant cross section many types of channel transitions are involved by hydraulic structures like gates, inlets, outlets and so on. Although there is a variety of hydraulic functions in service, the general classification of channel transitions are described in the following, by A.T. Ippen[11].

(1) Dissipation of energy, as by drop structures, spillway buckets, and stilling pools.

(2) Reduction of velocities to prevent scour, as in irrigation canals, or increase of velocities to prevent shoaling, as in navigation canals.

(3) Change in channel section or alignment with a minimum of energy dissipation as in power developments.

Further hydraulic function of channel transitions is to measure discharge, as by Venturi flumes, whereas the metering of discharge is commonly made by control structures as have been described in several sections.

Like all engineering structures, the design procedures are the try and cut methods until the most economical combination of objectives in hydraulic works is found that satisfies the basic requirements in various fields. However, in hydraulic professions of design procedure, the design problem of channel transitions is basically to estimate the surface profiles of water for design discharges. The calculation method of surface profiles which has been discussed in the foregoing part will be generally an available mean for fulfillment to this purpose. For hydraulic design of transitions for tranquil flows, the hydraulic performance of the structure will be estimated from downstream end, while the performance for transitions for shooting flows from the upstream end. As special types of channel transitions, often used are the discontinuities in channel geometry and grade like channel contraction and expansion, postive and negative steps. The channel contraction and expansion are used for reduction and increase of velocities of flows and positive and negative steps are for energy dissipation. These types of channel transitions are not evidently estimated owing to the discontinuity in channel section and grade. The model study will provide a satisfactory solution in hydraulic design of these types of transitions, and consequently a brief consideration to the performance of these transitions will be discussed in the present section. In the channel transitions for highly rapid flow, a series of surface disturbances called shock waves becomes from changes in wall alignment, with results of formation of standing waves. The freeboard of channel walls must be carefully estimated.

(1) Discontinuities in bottom grade known as positive and negative steps

Positive and negative steps are often constructed in hydraulic works of transition structures to dissipate the surplus amount of excessive energy and control the downstream water stage. The hydraulic performance of these steps are typical examples of a rapidly varied case suitable for treatment by means of the momentum theorem as cited by C. Jaeger[12].

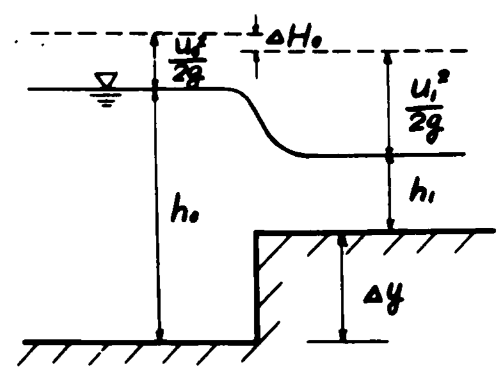

Fig. 4-2 Flows over positive step

H.A. Doeringsfeld and C.L. Barker[13] found experimentally the pressure was essentially hydrostatic at the upstream face of positive step and expressed in a form of

$$\int_0^{\Delta y}(p/\rho g)dy = (2h_o - \Delta y)\Delta y/2. \tag{4}$$

With the use of Eq.(4), the momentum conservation law is

$$(gh_o^2/2) + (q^2/h_o) = (gh_1^2/2) + (q^2/h_1) + (g/2)(2h_o - \Delta y)\Delta y + \zeta(q^2/2h_o^2)\Delta y. \tag{5}$$

Introducing the dimensionless parameters of $F_o^2 = (q^2/gh_o^3)$, $(\Delta y/h_o) = \zeta$, and $(h_1/h_o) = z$, Eq. (5) becomes

$$(1/2) + F_o^2 = (z^2/2) + (F_o^2/z) + (2 - \delta)(\delta/2) + \zeta(F_o^2/2)\delta . \tag{6}$$

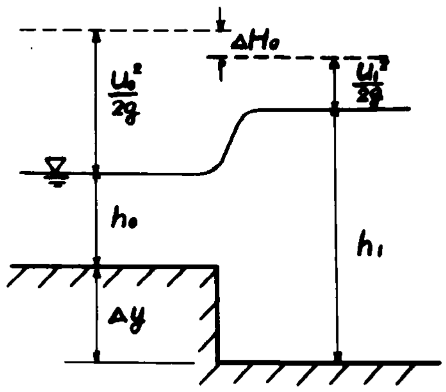

Fig. 4-3 Flows over negative step

In Eq,(6), ζ as the coefficient for form resistance varies from negative to pssitive depending on the geometrical form of discontinuities. As a first approximation in practical procedures of numerical calculation, the term containing is of only secondary importance and especially for streamlined form, so that ignoring the last term in the right hand of Eq.(6), the upstream Froude number is expressible as functions of z and δ .

$$F_o^2 = z(z - \delta + 1)(z + \delta - 1)/2(z - 1). \tag{7}$$

Fig. 4-4 indicates the relationship of Eq.(7), graphically, and for positive steps, two areas of (I) and (II) enclosed by two curves of $\delta = 0$ and $z = 1$ are possible domains. The area (I) indicates the upstream flow is tranquil, whereas the downstream flow is shooting or tranquil. The line $z^3 = F_o^2$ divides the area into two parts of tranquil and shooting. The right hand area illustrates a transition from tranquil to

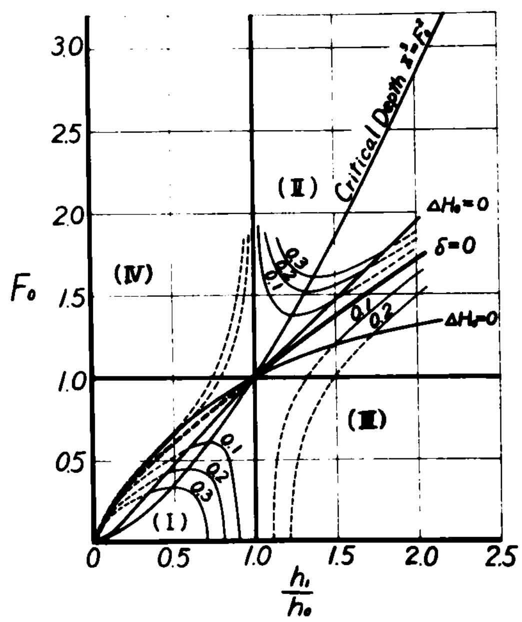

Fig. 4-4 Relations between upstream flow and change of fepth through discontinuities

to another tranquil flow, whereas the left hand area transition from tranquil to shooting and thus the positive step structure is classified as a channel control in the latter case. When the flow passes through discontinuities, the energy can not be gained, so that the energy dissipation rate is

$$(\Delta H_o/h_o) = 1 - \delta - z + (F_o{}^2/2z^2)(z^2 - 1) \geqq 0, \qquad (8)$$

and the flow can not be actually occured when the above inequality is not satisfied. The most satisfactory case for the energy dissipation is $\Delta H_o = 0$. Consequently, in Fig. 4-4 the dotted area describes the domain in which the flow can never occur in reality. Fig. 4-5 also indicates the relationship between z and δ as a parametric expression of F_o. The area (II) indicates the upstream flow is shooting and two transitions from shooting to tranquil and from shooting to shooting will be occured.

When the flow regime is same between up- and downstream reaches, positive step structure may be classified as a channel transition and the surface profiles of water will be readily calculated by means of the usual procedure and the preceding relationship. On the other hand, when the flow regime changes through the step, the structure is a channel control for upstream tranquil flows and a smooth transition from tranquil to shooting flow will be occured. As have been discussed in the fore-

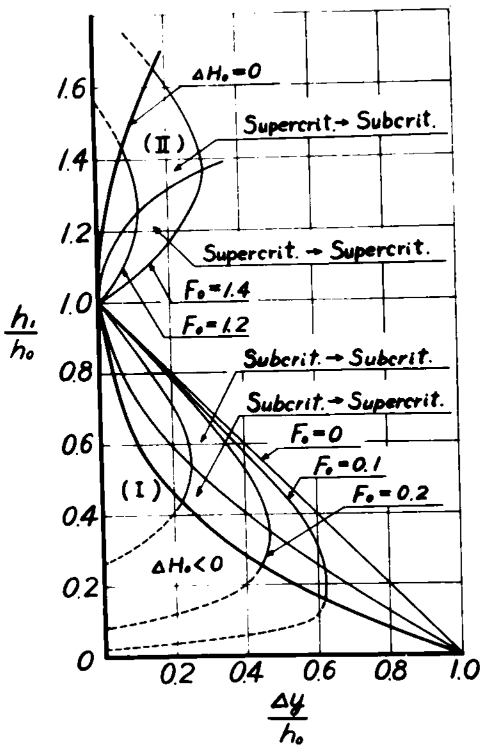

Fig. 4-5 Relations between z and for discontinuities

going parts, the transition from tranquil to shooting is only possible by the saddle point. At the positive step, the curvilinear influence becomes appreciable and the flow is not paralleled as treated in this subsection. If the basic flow characteristics for surface profiles are exactly expressed as a complete mathematical form, the hydraulic performance of discontinuities will be advanced. For given rate of discharge and channel geometry, two downstream values of sub- and supercritical are resulted. The possible downstream water depth will be practically determined by the downstream channel condition, and if the transition for supercritical flows is produced, the hydraulic jump will be followed in forms of normal, undular and submerged.

In the same manner, the hydraulic performance of negative discontinuities can be treated. Assuming the pressure distribution is similar to the case of positive step, the dimensionless momentum equation ignored the form resistance is

$$(1/2) + F_o^2 + (2z - \delta)(\delta/2) = (z^2/2) + (F_o^2/z), \qquad (9)$$

and solving Eq.(9) for F_o is

$$F_o^2 = z(z - \delta - 1)(z - \delta + 1)/2(z - 1). \qquad (10)$$

Fig. 4-4 illustrates the behaviours of F_o for various values of z and δ. The area (III) indicates the downstream depth of water increases when the flow passes through the negative step. Considering the energy dissipation is positive and

$$(\Delta H_o/h_o) = 1 + \delta - z + (F_o^2/2z^2)(z^2 - 1) \geqq 0, \qquad (11)$$

the solid lined area is the possible domain for actual flows. The flow regime changes from shooting to tranquil with a formation of hydraulic jump. If the downstream depth is large, the downstream influence will be travelled far upstream, and the careful treatment for the hydraulic behaviours of various types of jump must be accented.

The area (IV) is the case in which the downstream depth decreases. The possible hydraulic performance of negative step is very restricted in a small area, and the transition from tranquil to shooting will be seen, though the figure can not make visible owing to very small possible area. Actually, the height of step Δy must not be too great.

The preceding treatment of hydraulic performance of positive and negative steps in open channels is essentially based on the approximation of paralled flows, and the exact behaviors are not concerned, owing to greatly complicated phenomena of rapidly varied flows which have not been described in a mathematical form. The most available information is only derived by the model study and with combination of the theoretical estimation expressed in the foregoing, the hydraulic design of discontinuities in channels will be possible. If the steps are of stream-lined form, and the channel grade and cross section are continuously changed, the general theory of transitional characteristics of open channel flows will be applied.

The classical theory for discharge estimation by a broad crested weir is implied by the hydraulic performance of positive steps. The flow changes from tranquil in the upstream reservoir to shooting over the weir and thus the critical condition must be occured at a point never definitely determined by the classical theory. Fig. 3-10 illustrates changes of the energy coreection coefficient α and the pressure correction factor of Jaeger λ in the flow over a long weir, which is of dimensions of 4.00 m in length and 0.40 m in width. In the inlet portion, α becomes greater than unity, while λ becomes less than unity, which indicates the curvilinear flow. The same tendency in λ is also seen at the free overfall section, whereas the behaviour of α at this section is nearly the same as in the middle portion of a weir. The hydraulic performance of broad crested weir will be therefore developed by means of the theory of transitional characteristics described in Part II after hydraulic variables of α and λ are completely descriptive in a mathematical form in terms of other variables of flow characteristics which can be measured.

(2) Hydraulic performance of channel contraction and expansion

Channel contractions and expansions are frequently constructed in hydraulic works for various purposes of use. Hydraulic performance of channel contractions and expansions are similarly treated in the same manner as in discontinuities of bottom grades by means of the momentum conservation law and energy dissipation rate, as a first approximation. When such transitions in cross section are of gradual change, the calculation may be made, after dividing the whole transition into a series of several small discontinuities in cross section. For each discontinuity divided, the same chart as in the foregoing case will be readily provided for the hydraulic design of contraction and expansion. Four types of transitional characteristics, from tranquil to tranquil, from tranquil to shooting, from shooting to tranquil, and from shooting to shooting will occur. In the hydraulic design of conveyance structures, the most careful treatment must be considered to the tranquil transitions, because the basic principles of design for this type of transitions is to ensure the minimum energy dissipation consistent with economy ofconstruction. Another important class of transition is that for shooting transitions which carries the high velocity flows, because the formation of discontinuous wave front known as shock waves is resulted, and it will be briefly described in the later.

The preceding procedure by means of a simple expression in momentum conservation law can not involve the energy dissipation by the boundary resistance. In reality, the gradually varied flows in gradually changed transitions are extremely influenced by the action of boundary resistance, so that the details of flow pattern in channel transitions must be calculated by the computation procedure for surface profiles of water. The other significant factor which should be taken into consideration is the form resistance. Abrupt changes of cross section in channel contractions and expansions produce an appreciable magnitude of turbulence. The one dimensional procedures of approach in hydraulics of open channel flows are then not exactly valid, as the actual flow pattern in the transition is extremely complicated compared with the usual one in open channels, and furthermore the two dimensional turbulent influences known as the form resistance can not be expressed in the dynamic relationship. The model study for a particular design in channel transitions is only a suitable mean, and commonly used, though the approximate procedure to estimate the surface profiles in channel transitions may be treated by means of the introduction of hydraulics of jet diffusion. Apparently, the pertinent design for subcritical transitions is made so streamlined as to be a minimum lead loss. The recommendations of J. Hinds[14)] and F.C. Scobey[15)] for design criteria of channel transitions are that for converging and diverging transitions, the angle between the channel axis and the wall is less than 12.5 degree, and among three types of channel transitions, cylinder-quadrant, quadrant, wedge-type and warped, the best hydraulic performance is obtained in the last type of transition as easily expected.

When the shape transformation in channel transitions is so gradual as to be practically ignored, the comprehensive hydraulic performance of transition structures is theoretically calculated by means of the theory of transitional characteristics of gradually varied flows, which will be again described in the last subsection.

(3) Two dimensional transitions for shooting flows

The shooting flows are characterized by many outstanding hydraulic behaviours, among which the shock wave is of special evidence. The problem of two dimensional transitions for supercritical flows has been studied by many investigators, who used commonly the analogous theory between hydrodynamics and gas dynamics developed since the first decade of this century, and Ippen[16)] and his colleagues have been enforced to make the formulation of hydraulic performance in such transitions. However, the basic flow characteristics being similar to various types of hydraulic jump are so complicated that the complete non-linear behaviours can not explicitely be established. Recent works of J.J. Stoker[17)] on the shock wave are exactly mathematical and therefore the advance in this field considered to the hydraulic evidence in the flow pattern is urgently needed. The design criteria, however, for the transitions are seen in the literature of Ippen and others and the determination of channel walls in transition structures is approximately made with the use of charts described in literatures.

(4) Design procedures of channel transitions

In the foregoing, hydraulic performances of special types in channel transitions involving controls are treated. The basic principles for hydraulic design of conveyance structures are to obtain a minimum of energy dissipation consistent with economy of construction in a canal, so that the desirable design is furnished with the best feature selected through a large number of try and cut methods for tentative schedules. The exact evaluation of surface profiles of water is of essentail significance for tentative channel characteristics. Consequently, the theory of transitional characteristics of steady flows become important. In usual computation procedures, the method of numerical analysis is common, though the rough estimation of surface profiles of water in channels can be made by methods of Silber and others. If the theory of transitional characteristics in natural channels is carefully applied to the computation procedure of numerical analysis, much errors are not involved in the solution. This description will be illustrated in the following example[18].

When the discharge of 1 m^3/sec is carried in a divergent channel, in which the channel width is $(1 + 0.001x^2)$ m, the Chézy roughness is $\sqrt{1960}$ (m-sec), and the bed slope is 0.0025, the surface profiles of water will be calculated. For the sake of simplicity in numerical calculation, the shear friction will be two-dimensional and thus the hydraulic radius will be approximated by the water depth. Under such assumptions, the critical slope is 0.0050.

The first singular point is located at x_c = 31.274 m and h_c = 0.4391 m, and the second at x_c = 122.140 m and h_c = 0.2550 m. At the first one, it is classified as saddle, whereas the second becomes nodal. The slope of transition curve at the saddle point is -0.00297 and that of the other singular solution is 0.00464. All surface profiles near the nodal point approach it with the surface slope of 0.00129. The construction of possible surface profiles for the given

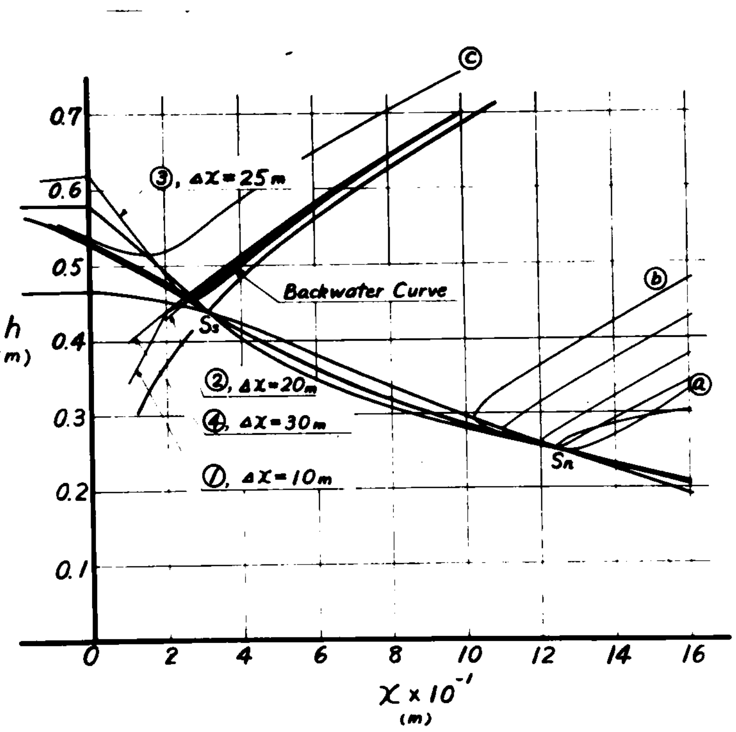

Fig. 4-6 Surface profiles of water in divergent channels

discharge is obtained by connecting two curves calculated up- and downstream from the saddle point, when no other control structures exist. Fig. 4-6 indicates the transition curve with other possible surface profiles. In the figure, the curve a indicates that the flow involves two control sections classified as saddle and nodal in the geometric theory of differential equation. Although the nodal point may be occasionally a control section, the range of possibility is very limited, so that in common channels, the control section classified as nodal is rarely happened in natural channels. When the downstream stage is lifted by other control structures like gates and dams, the curve b will be seen in channels. In this case, the transition curve calculated downstream from the saddle point and the downstream surface profiles calculated from the downstream control are connected through the hydraulic jump at the point where both depths of surface profiles become conjugate in the momentum conservation law.

The curve c illustrates the surface profiles produced by large submergences. The saddle point is no longer a transitional point in this case, and the surface profiles are estimated from the downstream control, and called usually backwater curves. However, in many practical calculations, these considerations to the transitional characteristics are often ignored, and the resulting surface profiles of water indicate no hydraulic significance of the determination of hydraulic design. If the water surface is regulated at 0.70 m in depth at the point of x = 100 m in the figure, the surface profile is apparently a back water curve which must be traced from the downstream end underinvestigation. If the calculation procedure is made by the finite difference method, the distance interval must be carefully selected. Fig. 4-7 indicates calcuated surface profiles for various distance intervals of Δx = 10, 20, 25 and 30 m. At points far from the singular point, no appreciable influences on the

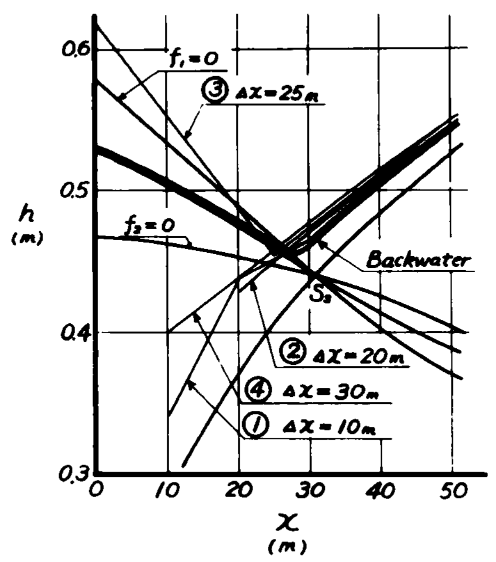

Fig. 4-7 Details of surface profiles of water near the saddle point

calculation are not observed, whereas in the very vicinity of the point, the selection of distance intervals becomes significant. For curves of ①, ②, and ④, the water depth decreases rapidly near the saddle point, and no desirable solutions are obtained. If 25 m is selected as the distance interval in this case, the resulting surface profile is closely related to the exact solution, whereas the detail pattern of flow is distinctively different from the back water curve.

As will be recognized in the preceding example, the surface profiles of water in natural channels must be carefully traced and especially in the immediate vicinity of singular points. If not, the numerical analysis with smaller intervals in distance can not express the hydraulic significance in the resulting solution, and this treatment is essentially important to the establishment of pondage area of a reservoir in mountaneous district, and it may be said no speical considerations to this theory of transitional characteristics have been accented in actual design procedures. The successful analysis for correct determination of surface profiles must be followed in the description indicated in 2-4-4, and the more data in basic channel characteristics are provided, the better conclusion in surface profiles is obtained.

When the basic surface profiles of water for the design discharge is determined in a hydraulic project under investigation, the necessary channel wall is also estimated by adding a sufficient height of freeboard. If the basic computation in hydraulic researches of a project is completely exact, no freeboard is required, as the water surface is accurately anticipated during the preliminary design. However, the present hydraulic researches in open channel flows have not been fully developed in the detail pattern of flows, so that secondary influences are frequently observed in the channel. Additional freeboard will, therefore, provide the safety-carriage of water in a channel. For channels, which carry the sudden increase or decrease of discharge, the hydraulic knowledge of unsteady flows must be also considered.

---

1) Houk, I.E., Irrigation Engineering, Vol. 2, Projects, Conduits, and Structures, John Wiley, New York, 1956.

2) Kandaswamy, P.K., and Rouse, H., Characteristics of Flow over Terminal Weirs and Sills, Jour. Hydraulics Division, Proc. ASCE, HY 4, Aug. 1957.

3) Iwagaki, Y., and Iwasa, Y., On the Hydraulic Characteristics of the Roll-Wave Trains, - Studies on the Thin Sheet Flow, 7 th Report -, Jour. JSCE, Vol. 40, No. 1, Jan. 1955 (in Japanese).

4) Lane, E.A., Recent Studies on Flow Conditions in Steep Chutes, Engineering News Record, Jan. 2, 1936.

5) Halbronn, G., Étude de la mise en régime des écoulements sur les ouvrages à forte pente, La Houille Blanche, Jan.-Feb. 1952.

6) Craya, A., and Delleur, J.W., An Analysis of Boundary Layer Growth in Open Conduits near Critical Regime, Dept. of Civil Eng., Columbia University, CU-1-52-ONR-266, 1952.

7) Bauer, W.J., Turbulent Boundary Layer on Steep Slopes, Trans. ASCE, 1954.

8) Michels, V., and Lovely, M., Some Prototype Observations of Air Entrained Flow, Proc. Minnesota Int. Hyd. Convention, Sept. 1953.

9) Gumensky, D.B., Air Entrained in Fast Water Affects Design of Training Walls and Stilling Basins, Civil Engineering, Dec. 1949.

10) Hall, L.S., Open Channel Flow at High Velocities, Proc. ASCE, Sept. 1943.

11) Ippen, A.T., Channel Transitions and Controls, Engineering Hydraulics, edited by H. Rouse, John Wiley, New York, 1950.

12) Jaeger, C., Engineering Fluid Mechanics, Blackie, London, 1956.

13) Doeringsfeld, H.A., and Barker, C.L., Pressure-Momentum Theory Applied to the Broad Crested Weir, Proc. ASCE, Vol. 65, 1939.

14) Hinds, J., The Hydraulic Design of Flumes and Siphon Transitions, Trans. ASCE, Vol. 92, 1928.

15) Scobey, F.C., The Flow of Water in Flumes, U.S. Department of Agriculture, Tech. Bull. 393, Dec. 1933.

16) Ippen, A.T., Mechanics of Supercritical Flow, Proc. ASCE, Nov. 1949.

17) Stoker, J.J., The Formation of Breakers and Bores, Comm. Appl. Mech., Vol. 1, No. 1, 1948.

18) Iwasa, Y., Theoretical Study of Hydraulic Behaviours of Boundary Charactersitcs to Channel Transitions and Controls in Divergent or Convergent Channels, Trans. JSCE, No. 59, Separate 3-1, Nov. 1958.

2. Hydraulic Design of Control Structures

As has been briefly described in the foregoing part, a control structure is the class of transition structures in which the flow provides the control section for carried discharges, and consequently the hydraulic behaviours of flow in control structures are completely determined by means of the theory of control section. Control structures are used for various purposes in hydraulic projects and especially the most available use is to measure the discharge of flow by control structures. With respect to this use, various types of weir and flume are so familiar to hydraulic engineers. Additional use of control structures is to exclude heavy sediments, as the flow regime changes from tranquil to shooting in the structure.

Conveyance structures become channel controls under some hydraulic conditions, though in the original project they are not planned as controls, and in almost all cases, conveyance structures involve channel controls locally. Therefore, the purpose is very wide and the hydraulic design for control structures can not be unified.

Usually, control structures may be divided into two classes of uncontrolled and controlled. The controlled class is the structure which is constructed by additional appurtenant structures in canals and can control the flow state by the artificial regulations. Gates are typical examples of this class. When the conveyance structures involve local changes in channel geometry like cross section and channel grade as well as boundary resistance, the flow changes its regime in the structure, and for particular discharges the structure may become channel controls and be classified as uncontrolled class. Consequently, a spillway crest, a weir and others are examples of this class. The flow behaviours in controlling structures are generally so complicated as not to be expressed in a mathematical description of one dimensional procedures in hydraulics. Common procedures are the model study for particular designs. The complete formulation of this type of structures is one of further research problems in terms of modern knowledge in hydrodynamics and hydraulics.

In this chapter, therefore, the uncontrolled class of control structures is exclusively treated. As described, the hydraulic performance of this type of structure is definitely determined by means of the theory of transitional characteristics of steady flows, and it is apparently observed that the hydraulic design of control structures must be projected by a large number of requirements involving in the work. Hydraulically speaking, the flow in control structures is also classified into two parts of gradually and rapidly varied flows, which are frequently described in the foregoing parts. The gradually varied flows are essentially characterized by the hydrostatic law in pressure distribution and the negligence of surface curvature at the free surface, as the flow is gradually changing in channel transitions and controls. On the contrary, the rapidly varied flow is mainly characterized by the rapid change in flow behaviours within a relatively short passage through the structure. Consequently, in many cases treating with the hydraulic behaviours of rapidly varied flows, the detail pattern of flow itself can not be subject in a mathematical form, which implies the hydraulic analysis by means of the one dimensional method is insufficient to make the hydraulic performance of the structure clear. Often used is the momentum approach which is constructed within a zone involving such control sections, as seen in the previous chapter and many hydraulic literatures. Nevertheless, it is evident that this method can not indicate definitely the transitional characteristics and the hydraulic performance of structures, so that in classical theories on hydraulics of controlling performance many empirical coefficients have been introduced as a mean to estimate the behaviors in more refined forms.

For hydraulic design of flumes and other control structures through which the gradually varied flow is carried, the hydraulic requirement is rather simple. The change of flow behaviours in structures is so gradual that the sudden increase or decrease in velocity and pressure is not locally occured. In this case, the exact measurement of discharge by the control structure is only the necessary requirement. The structures must be designed and constructed so carefully as to be consistent with the theory of control section for gradually varied flows described in the foregoing part. In reality, the more desirable design criteria are not established. Among a large variety of tentative schedules imposed by topographical and geological conditions as well as economy, the design suitable to the project site must be selected by the try and cut procedure. Flumes, as a simple mean for discharge measurement, have not yet completely standardized, and various types of flumes like those of Parshall, Inglis, de Marchi and others are still widely used in the world. A tremendous amount of experimental works and theoretical analyses for various types of flumes combined with two variables of cross section and channel grade will furnish the final model of the most favourable design for discharge metering by a single water-level measurement, and the settlement of this problem is one of further program for hydraulic engineers who are participating to the hydraulics of open channel flows.

On the other hand, hydraulic designs of weirs, overflow spillways and other control structures, through which the rapidly varied flow is carried, are required by the two requirements of control section and pressure distribution. The former requirement is apparently the same one as the designs of flumes, because the structure must be involved by control sections to insure the discharge measurement. Herewith, the other difficulty for hydraulic analysis on performance of structure will be arised, as the rapidly varied flow changes the flow state within a short passage, and frequently the detail pattern of flow behaviours required for the hydraulic analysis and thus the design procedure can not be expressed in a mathematical form. The flow over a round crested weir and a spillway crest is practically expressed by a constancy of angular velocity as described in the foregoing part with sufficient accuracy for engineering purposes, so that the design procedure will also provided only by the theoretical side, whereas the flow over a sharp crested weir is more complicated in its dynamic principles and consequently the clear formulation of head-discharge relationship as the fundamental principles of control structures becomes difficult in terms of the one dimensional hydraulic procedures.

For short structures like a round crested weir and a spillway crest, the most important requirement for hydraulic design is the pressure distribution along the boundary of structure. The flow on a short structure is evidently very rapid in the change of flow state, so that the rapid increase of velocity makes consequently the sudden decrease of pressure and in some cases, it is a cause of the formation of cavity along the boundary and unfavourable vibration of the structure. The cost of repair and maintenance of the structure will certainly be raised. The investigation of cavity formations in open channels has not been progressed compared with other field of hydraulic machinery. The present design recommendation for prevention from the cavitation is to provide the channel geometry in forms of so streamlined as not to produce the local decrease and increase of pressure. The author[1)] and his colleagues attempted to measure the pressure fluctuation in fluid flows over a curved boundary with the use of pressure gauge. The formation of cavity has not been definitely detected. However, being different from the turbulent fluctuation in fluid flows, the possibility of cavity formation was apparently indicated with results of recording the pressure fluctuation of several cycles. At downstream faces of overflow spillways, it is frequently observed that the cavity formation produces the scaling of concrete. It is then recognized that the design procedure must be accented to the exact estimation of pressure distribution on the solid surface. From this point of view, the design procedure of short structures in pressure distribution will be briefly described in the follwoing, with some examples.

(1) Pressure requirement for round crested weir

The pressure distribution along the curved crest of weirs and the locations where the bed pressure becomes zero has already been discussed in Part III. The experimental results also verified the theoretical calculations derived by the irrotational theory, and the design criteria for necessary length of curved boundary are indicated in Fig. 3-5. If the range of discharge which is carried over the weir is determined during the preliminary project, the required length is readily obtainable in the figure.

In actual behaviours of flows over the weir, it is commonly observed by the experimental study that the unfavourable pressure decrease will occur at the upstream face, where the flow being initially at rest obtains an extreme value in velocity. In this reach, the present procedure by means of the irrotational theory can not express the flow pattern with sufficient accuracy, and therefore the transition curve from the upstream face to the weir will be recommended.

(2) Some comments to design procedure of control structure at overflow spillway

The hydraulic design of crest shape of control structure at the crest of overflow spillway is mainly based on the pressure requirement, which is to prevent the structure face from the extreme decrease in pressure due to rapid increase of velocity. Consequently, the common method to determine the crest shape is to use the experimental results of Creager for the nappe of free flow. The crest shape designed by the curve of Creager is then called as the standard crest. At the lower nappe of free flow it is evidently atmospheric, so that the standard curve is the favourable shape from the point view of pressure requirement. However, the standard curve of Creager can not be described in a complete mathematical form, and much endeavours to obtain the formula of free nappe are still continued. When the co-ordinates are, therefore, required for hydraulic design of crest shape, the composite curves, each of which is locally fitted to the standard curve of Creager, is used, and commonly the upstream curve and the downstream curve are connected at the crest of control structure. In this case, as has been described in the preceding part, the crest shape becomes discontinuous in local values of curvature. The exact estimation of head-discharge characteristics by means of the theory of transitional characteristics is not possible, and therefore, the theoretical verification of discharge relationship to the experimental results of scale model tests can not be made. To obtain the satisfactory estimation for the head-discharge relationship by overflow spillways, several model tests must be conducted. For hydraulic design of crest shape at control structures of overflow spillways supported by the experimental results of scale model tests and the theoretical verification, it will be favourable that the pressure requirement and the discharge characteristics must be equally taken into consideration when the design procedure is in progress.

---

1) Ishihara, T., Iwasa, Y., and Kinukawa, S., Measurement of Pressure Fluctuations in Open Channel Flows, Report, 3 rd Sym. Hyd. Res., JSCE, May 1958 (in Japanese).

CONCLUSIVE STATEMENT

The hydraulic designs of conveyance and control structures involved in the hydraulic works as one of gravity projects have been steadily progressed with the advance in basic knowledge of hydraulics and hydrodynamics. In the oldest days, the theoretical background for engineering project was essentially based on the theory of uniform flows. Since the works of scientists and engineers in the last centruy, the theory of non-uniform flow became familiar to hydraulic engineers as a mean for hydraulic design with conducting the scale model tests. With the improvement of hydraulic works and the basic theory, the more exact design procedures will be expected. In view of this requirement, the present paper concerns with the design procedures in hydraulic structures, mainly in conveyance and control structures, in the light of modern knowledge of hydraulics and applied mathematics recently developed. The complete description for recommendation of hydraulic design criteria is not obtained because of great complexity in flow behaviours characterized by turbulence and non-linear influences. Nevertheless, it will be expected that the study described in this paper gives some available informations on the real achievement in hydraulic designs for conveyance and control structures to some extent.

As the conclusive statement of the study, the followings are summarized.

In Part I, the general hydraulic characters of open channel flows are concerned by means of the one dimensional procedure of hydraulic analysis. This procedure was initiated in the 19 th century, and, in the present day, it is still very familiar method to describe the whole pattern of flow for hydraulic engineers. The whole study in this paper is treated by the one dimensional procedure.

The first chapter describes the basic dynamics of open channel flows so detailedly as to be favourable for the further discussion of transitional and transient problems of open channel flows. In this chapter, the rather new treatment for hydraulic researches is the introduction of boundary layer theory to the flow in open channels. Compared with the successful development of the boundary layer theory in other fields of aerodynamics, meteorology and so on, the progress of this theory in the hydraulics of open channel flows is still slow as the experimentations are so difficult that the experimental verification can not be solved. Nevertheless, with the use of this theory, the formation of free surface disturbances and air-entrainment can be practically estimated in a mathematical form. Many other problems which will be solved by the boundary layer theory are existed in hydraulics, and with respect to this statement, the introduction of boundary layer theory to open channel flows is of great significance for the further progress in hydraulic researches.

In the second chapter, the general theory of transitional characteristics of steady flows is described by means of the geometric theory of ordinary differential equation. This theory provides the most significant relationship for hydraulic designs of conveyance and control structures, and in following parts, the design procedures and the hydraulic performances of structures are discussed in terms of the transitional characteristics of steady flows, as the hydraulics of steady flows is the basic knowledge of design procedures. Therefore, this chapter is the fundamental essential of the whole study described in this paper.

The transitional characteristics of steady flows are conclusively described in the following way:

The change of flow state, from tranquil to shooting or vice versa, is produced by the existence of singular point in the basic equation of surface profiles expressed in terms of one dimensional procedure for hydraulic research. The transition from tranquil to shooting occurs at the saddle point, whereas that from shooting to tranquil is occured by nodal and focal points, generally through the hydraulic jump. The classifications of three kinds of singular points are definitely determined by the discharge and channel geometry in cross section and channel grade as well as channel roughness. All hydraulic characteristics of steady flows are exclusively obtained by the theory of transitional characteristics. In gradually varied flows, the exact estimation of surface profiles of water can be possible with the use of this theory and in rapidly varied flows the hydraulics of control section is completely established and therefore the head-discharge relationship for discharge measurement by a single water-level observation is also provided.

Consequently, the theory described herein is the generalized theory of steady flows in hydraulics developed since the works of Dupuit and Bresse, and the transitional theory of steady flows will certainly become the basic knowledge of further hydraulic researches.

In relation to the general theory of transitional characteristics of steady flows in channel transitions and controls, the most significant theorem on simultaneity of maximum discharge, minimum energy and minimum momentum flux known as the generalized theory of Jaeger was established at the saddle point by means of the transitional theory. This theorem can substantially indicate the head-discharge relationship of control structures and determine the design procedure of control structures. In other words, it is a solution of the philosophical question why the flow can only pass the channel with the least rate of energy dissipation.

Another important conclusion in this chapter is the discussion of initiation condition for hydraulic instability which indicates the transition from one uniform state of smooth plane flow to another state of rough surface. Although the Vedernikov criterion on initial instability is famous for hydraulic engineers and verified by many engineers including the author, the higher approximation indicates the formation of instability in the flow depends on the dispersive property of fluid flows in open channels. The further experimental verification will indicate the detail pattern of flow and therefore the instability formation in open channel flows.

The third chapter concerns with the hydraulic characteristics of transient problems of open channel flows. Particularly in the present study, the transient problems are solved by the one dimensional procedures of analysis and the theory of moving discontinuity. The first order theory of surges and translation waves have been completely developed by a large number of scientists and engineers. Consequently, the accent of the study was directed to the second order theory characterized by the appreciable magnitude in vertical acceleration. The resulting solution is expressed in forms of cnoidal and solitary waves. Although the hydraulic characteristics of cnoidal and solitary waves are not explicitely clarified, the wave form continues without changing its initial form, so that the hydraulic design for conveyance and similar structures in which the increase and decrease in discharge is resulted by sudden changes in load must be carefully treated.

In Part II, the hydraulics of gradually varied flows in uniform and non-uniform channels like channel transitions and controls is concerned, and especially the transitional characteristics of gradually varied flows are completely expressed. The gradually varied flows are mainly characterized by the hydrostatic pressure distribution and the negligence of surface curvature, because the flow changes gradually in the course of channels. The hydraulics of gradually varied flows in uniform channels has been developed since the works of Dupuit and Bresse in the 19 th century and in the present day the results studied by many scientists and engineers are quite outstanding. In reality, the fruitful tabulations for back water functions calculated by the tremendous amount of labours have been obtained.

Chapter 1 of this part describes a brief summary of such studies, and furthermore the influence of non-hydrostatic flows with appreciable vertical acceleration which become predominant near the critical depth is also concerned by means of the same mathematical procedures used for classification of transitional characteristics. The results indicate the flow becomes of wave form and without waves, depending on the channel characteristics, as have been described in the works of Boussinesq and Fawer.

Chapter 2 concerns with the summary of hydraulics of gradually varied flows in non-uniform channels developed since the 19 th century. Almost all natural channels and artifical watercourses are commonly non-uniform, so that the hydraulic behaviours of flows in such channels are more important for practical uses in engineering problems. Usual procedures of surface profile routing in non-uniform channels are a numerical analysis and a graphical method.

In non-uniform channels, however, both curves of critical and normal depths are not constant but a variable as functions of channel characteristics and distance for a particular value of discharge, so that the resulting surface profiles of water calculated by the procedure of numerical analysis and the graphical method are largely errorneous in the immediate vicinity of singular points, as the flow will be frequently changed in its state. The complete description of such behaviours will be illustrated in Chaperr 4.

In Chapter 3, the conclusive statement of various types of hydraulic jumps is presented. The hydraulics of jump phenomenon of normal type has been completely studied and the relationship for formation also has been established. Therefore the present attention is mainly accented to the hydraulic behaviours of undular jump characterized by a series of cnoidal and solitary forms. Nevertheless, the complete description of behaviours has not been obtained because the basic physics of such flow behaviours as the second approximation of non-linear open channel flows are not completely understood as seen in the transient problems described in Chapter 3 of Part I.

Chapter 4 is the main subject of the whole part and concerns with the transitional characteristics of gradually varied flows. As described in the foregoing, the surface profile routing of water in non-uniform channels must be essentially considered by the transitional characteristics, to obtain the exact evaluation of profiles for various purposes in engineering problems. This chapter, especially, deals with the transitional characteristics of Chézy flows as the basic flow pattern of open channel flows. The resulting profiles are classified by nine saddle points, five nodal points, and four focal points, each of which is also determined by channel and flow characteristics. With the aid of all possible surface profiles of gradually varied flows in channel transitions and controls, the hydraulic behaviours of transitional characteristics, the hydraulic performance of control structures and the surface profile routing will be illustrated. It is evidently that the routing of surface profiles of water is frequently errorneous if the complete knowledge on transitional characteristics of gradually varied flows is not obtained by hydraulic engineers. Apparently the same treatment will be readily extended to the flow in which the Manning type of resistance is used.

Part III concerns with the hydraulic performance of control structures as metering devices by a single water-level measurement. The control structure which involves the control section determines all characteristics of transition flows and therefore it is used as discharge metering for open channel flows.

Chapter 2 of the part concerns with the hydraulics of round crested weir as an example of control structures. The hydraulic characteristics of flows over a weir and the hydraulic performances in the structure are definitely determined by the use of transitional theory of irrotational flows and the generalized theory on simultaneity of maximum discharge and minimum energy. The theoretical results also are verified by the experimentations conducted at the laboratories. The head-discharge relationship of round crested weir can be expressed as a function of ratio between the curvature of solid boundary and the upstream head. Therefore, the hydraulic design for round crested weir as flow metering devices will be possible.

In Chapter 3, the hydraulics of control structure of overflow spillways is concerned by means of the same treatment. The classical method to estimate the head-discharge relationship of overflow spillways was exclusively based on the scale model tests. In this study, the present treatment by the transitional theory of flows over control structures indicates the establishment of head-discharge relationship is also possible with sufficient results of engineering purposes.

In Chapter 4, the hydraulic performance of a sharp crested weir is indicated. This problem is one of the most difficult problems in hydraulic researches, because the basic flow pattern is not subject to a complete mathematical description, whereas the hydraulics of control section described herein indicates that the procedure of transitional characteristics is one of the steps for further hydraulic reseraches which will be developed in the future. The results theoretically calculated are closely agreed with the experimental data obtained by a large number of engineers, and in the last section of this chapter, some comments of hydraulics of free overfalls are described. The clear formulation of hydraulic performance of free overfalls is not derived, while the better illustration for the performance is also indicated by the present approach.

Chapter 5 deals with the hydraulic performance of flumes as metering devices. In contrast with the foregoing cases for controlling devices of rapidly varied flows, the flume is usually called as long structures in which the gradually varied flows are carried. When the channel contraction and expansion are constructed in the flume, the transition flow can be occured and thus the structure also can become control structures. The flumes of Parshall, Inglis, de Marchi and others are those structures. The model flumes indicated a good agreement between the theoretical calculation and the experimental data, and therefore the discharge measurement by the use of such flumes can be possible with sufficient accuracies. However, the establishment of empirical formulas derived by the theoretical calculation or the experimental results is not possible because the basic flow pattern of gradually varied flows in the flume is largely characterized by the non-linear influences of fluid motion. Only the possible way is the tabulation for discharge coefficents by the tremendous amount of calculation.

The last part of the study deals with the hydraulic requirements for hydraulic design of conveyance and control structures and the design procedures of these structures with the use of the foregoing descriptions as the conclusive summaries of the whole study.

The canal consists usually of uniform channels, transition structures and control structures. For hydraulic designs of uniform mild channels as conveyance structures, the usual method of calculation can be applied. However, the final requirement for designs is to obtain the minimum energy loss consistent with the economy of construction, so that the design procedures are substantially the try and cut method. For designs of steep channels, other significant behaviours of shooting flows must be considered. The formation of roll waves by the hydraulic instability, the entrainment of air bubble into the flow and the sudden increase of water depth occured by the oblique standing wave, when the channel is of two dimensional transition, are carefully treated in the preliminary projects. The hydraulic behaviours of these characteristics are completely studied in this paper and will certainly serve as the real contributions to the desirable plans.

When the channel transition is planned in hydraulic works, the complete knowledge of transitional characteristics of flow must be required. The theory of transitional characteristics of gradually varied flows is evidently indicated in the foregoing part, whereas that for rapidly varied flows is not completely established, because the detail pattern of flow is not subjected in a mathematical form. If the geometrical change of channel is so gradual, the hydraulic design can be theoretically analyzed. If not, the empirical coefficients for form resistance, with the scale model tests, must be introduced under the present knowledge of one dimensional hydraulic procedures. This problem is still not clarified and must be continued to solve as one of further problems.

If the channel transition is classified as the control structure, all the hydraulic characteristics can be definitely established, so that the hydraulic design will be possible for engineering purposes. In this case, the other necessary requirement for pressure will be arised as well as for the discharge, Especially, the pressure requirement is very important for hydraulic design of short structures, in which the flow is very rapidly changed, to prevent the structure from the unfavourable decrease of pressure and the formation of cavity along the solid boundary.

ACKNOWLEDGMENTS

The whole subject described in the present study is a conclusive statement for establishment of the hydraulics of transitional characteristics of open channel flows by means of the modern knowledge in theoretical advances of mathematical, hydrodynamic and hydraulic fields. The research program has been continuously progressed by the author with cooperation of other staffs and students particularly participating to hydraulics and hydraulic engineering in the Hydraulics Laboratory, Department of Civil Engineering, Kyoto University, under the supervision of Dr. Tojiro Ishihara for recent several years.

Deep acknowledgments of the author are made to Dr. Ishihara for his continuous encouragements and instructions to promote the present research programs. Also thanks are due to staffs and students in the laboratory for their earnest help in developing this program. Especially, it is of great pleasure of the author that names of Messrs. H. Matsunami, S. Kinukawa, K. Ihda, T. Kadoya, A. Murakami, R. Nihari, F. Nishihira, M. Matsubara, and Y. Inoue should be remained in the paper as the author's symbol of grateful appreciation for their assistance.